Problem Set #4 Solutions

Course 14.06 – Intermediate Applied Macroeconomics

Distributed: April 7, 2004

Due: Thursday, April 14, 2004 [in class]

1. **Learning by Doing (Romer's AK Model)**

This question is based on the Learning by Doing model discussed in the class notes and Romer's textbook (page 120). The economy is described as follows:

$$u(c_t) = \frac{c_t^{1-1/\theta}}{1-1/\theta}$$

$$Y_t = K_t^{\alpha}(h_t L_t)^{1-\alpha}, \quad k_{t+1} = (1-\delta)k_t + i_t, \quad h_t = bk_t, \; b > 0, \quad L_t = L \; \forall t$$

(a) **Describe how human capital, as given by h_t, accumulates in this economy. Do firms or individuals directly invest in improving the level of human capital? Or, is it simply a side effect of physical capital accumulation? Given your answer, do you believe that the social planner and decentralized competitive equilibrium will coincide in this model? (worth 2 points)**

The accumulation of human capital in this economy is simply a side effect of the accumulation of physical capital. The more physical capital per unit of labor, k_t, the more human capital per unit of labor. Thus, the more physical capital workers have, the more human capital they will accumulate by working with that capital. They learn more by working with more physical capital… i.e. learning by doing.

Since human capital is not the result of direct investment and is simply an externality from the creation of physical capital, we should not expect that the decentralized competitive equilibrium will coincide with the social planner problem. In fact, we should expect that an underinvestment in physical capital will occur in the decentralized competitive equilibrium.

(b) **Write out and solve the social planner's problem in this economy. (Assume that the discount factor is $\beta \in (0,1)$) (worth 2 points)**

a. **What is the growth rate of consumption in the economy?**
b. **What is the optimal savings rate? (Assume a linear savings function as done in the class notes)**

As usual, the resource constraint of the economy is given by:

$$c_t + k_{t+1} = k_t^{\alpha} h_t^{1-\alpha} + (1-\delta)k_t$$

Plugging in for $h_t = bk_t$, the resource constraint becomes:

$$c_t + k_{t+1} = \left(1 + b^{1-\alpha} - \delta\right) k_t$$

The social planner's problem can then be written as:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t. } c_t + k_{t+1} = \left(1 + b^{1-\alpha} - \delta\right) k_t \;\; \forall t$$

The lagrangian can be written as:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \mu_t \left[\left(1 + b^{1-\alpha} - \delta\right) k_t - c_t - k_{t+1}\right]$$

The FOCS are as follows:

$$\begin{aligned} c_t &: \; \beta^t u'(c_t) = \mu_t \\ k_{t+1} &: \; \mu_t = \mu_{t+1}\left(1 + b^{1-\alpha} - \delta\right) \end{aligned}$$

Combining, we have the Euler condition:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left(1 + b^{1-\alpha} - \delta\right)$$

Thus, the growth of consumption is given by:

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1 + b^{1-\alpha} - \delta\right)\right]^{\theta}$$

Assuming a linear savings rate, we have by definition:

$$\begin{aligned} c_t &= (1-s)\left(1 + b^{1-\alpha} - \delta\right) k_t \\ k_{t+1} &= s\left(1 + b^{1-\alpha} - \delta\right) k_t \end{aligned}$$

Using $\frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t}$, we can solve for s explicitly as:

$$s = \beta^{\theta}\left(1 + b^{1-\alpha} - \delta\right)^{\theta - 1}$$

(c) Solve for the competitive equilibrium in this economy. (Solve the model in the same fashion as done in the first half of the course. For simplicity, assume that the labor supply of individuals is exogenous and equal to 1. Moreover, assume that the borrowing constraint is never binding.) (worth 2 points)

a. Prove the household budget constraint can be written as:

$$c_t + k_{t+1} + b_{t+1} = w_t h_t + (1 + r_t - \delta) k_t + (1 + R_t) b_t$$

where b_t are government bonds that have return R_t

The household budget constraint can be written as:

$$c_t + i_t^k + i_t^b = w_t h_t + r_t k_t + R_t b_t$$

The accumulation equations are:

$$k_{t+1} = (1-\delta)k_t + i_t^k$$
$$b_{t+1} = b_t + i_t^b$$

Plugging these into the budget constraint, we have:

$$c_t + k_{t+1} + b_{t+1} = w_t h_t + (1 + r_t - \delta)k_t + (1 + R_t)b_t$$

b. What is the arbitrage condition between risk-free bonds and capital?

The arbitrage condition is given by $R_t = r_t - \delta$. Otherwise, individuals would prefer bonds over capital, or vice versa. Using this arbitrage condition, we know that individuals will not care how they allocate their savings between capital and bonds. The budget constraint can be rewritten as:

$$c_t + a_{t+1} = w_t h_t + (1 + R_t)a_t \quad \text{where } a_t = k_t + b_t$$

c. Use the FOCs of the household to find the Euler condition.

The household's maximization problem thus becomes:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t. } c_t + a_{t+1} = (1 + R_t)a_t \ \ \forall t$$

The FOCs for c_t and a_{t+1} will give us:

$$\frac{c_{t+1}}{c_t} = \left[\beta(1 + R_t)\right]^{\theta}$$

d. Using the firm's profit maximizing behavior, what is the equilibrium interest rate for physical capital and the equilibrium wage rate? (Remember, the firm takes human capital as exogenous).

The firm has the following maximization problem:

$$\max_{K,L} K_t^{\alpha}\left(h_t L_t\right)^{1-\alpha} - w_t h_t L_t - r_t K_t$$

The FOCS are given by:

$$r_t = \alpha k_t^{\alpha-1} h_t^{1-\alpha} = \alpha b^{1-\alpha}$$
$$w_t = (1-\alpha) k_t^{\alpha} h_t^{-\alpha} = (1-\alpha) b^{-\alpha}$$

e. **What is the growth rate of consumption?**

Using $h_t = bk_t$, we have r $r_t = \alpha b^{1-\alpha}\ \forall t$. Using our arbitrage condition, and plugging into the individual's Euler condition, we thus have:

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1+\alpha b^{1-\alpha} - \delta\right)\right]^{\theta}$$

(d) Compare the growth rate of the social planner's equilibrium from Part B to that of the decentralized equilibrium in Part C. How are they different, and why? **(worth 2 points)**

Clearly, the growth rate in the decentralized competitive equilibrium is lower. The reason for this is that competitive firms do not take into account that investing in more capital will have an externality in increasing the human capital stock. The social planner, however, takes into account this externality from physical capital when it chooses the optimal amount of physical capital. Hence, the decentralized equilibrium has a lower growth rate because firms under-invest in physical capital relative to a social planner.

(e) Now suppose that the government subsidizes the private cost of capital for firms. In particular, assume that the private cost of capital for firms is now given by $(1-\tau)r_t$ and the government pays for this subsidy via lump sum taxes, T, on individuals. Resolve for the competitive equilibrium of this economy. **(worth 2 points)**

The individual's budget constraint can now be written as:

$$c_t + k_{t+1} + b_{t+1} = w_t h_t + (1 + r_t - \delta) k_t + (1 + R_t) b_t - T_t$$

The arbitrage condition is the same as before, and the consumers FOCs will now be affected by the lump sum tax. Thus, we have the same Euler condition as before:

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1 + r_t - \delta\right)\right]^{\theta}$$

Firms, however, will now choose their physical capital stock such that:

$$(1-\tau) r_t = \alpha k_t^{\alpha-1} h_t^{1-\alpha}$$

Plugging in for $h_t = bk_t$, we have $r_t = \dfrac{\alpha b^{1-\alpha}}{1-\tau}$.

Using the arbitrage condition and plugging into the Euler condition, we have:

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1+\frac{\alpha b^{1-\alpha}}{1-\tau}-\delta\right)\right]^{\theta}$$

a. **What is the subsidy, τ, the government should set such that the competitive equilibrium growth rate of consumption coincides with the social planner's outcome we found in Part A?**

In order to return us to the equilibrium growth rate of the social planner's economy, the government should set $\tau = 1-\alpha$.

b. **Why does the government want to subsidize investment in this model?**

The government wishes to subsidize investment in physical capital since firms undervalue the return to capital since they do not take into account the additional learning by doing that can be done with more physical capital.

2. Taxes in the Human Capital Model

This question will add taxes to the human capital model discussed in the notes. Specifically, assume the following conditions for the economy:

$$u(c_t) = \frac{c_t^{1-1/\theta}}{1-1/\theta}$$

$$Y_t = BK_t^{\alpha}(h_t L_t)^{1-\beta}, \qquad B>0, \qquad L_t = L \ \forall t$$

$$k_{t+1} = (1-\delta)k_t + i_t^k$$

$$h_{t+1} = (1-\delta)h_t + i_t^h$$

(a) What must I assume about the variables α and β in order to have constant returns to scale (CRS) for physical capital and human capital? Why do we want to make this type of assumption? (worth 1 point)

In order to have CRS, it must be that $\beta = \alpha$. We want to make this assumption so that our model will generate a linear return to our joint investment in human and physical capital in the model. (See the class notes on this model for more information).

For the remainder of this question, assume CRS as done in Part A.

(b) Solve the social planner's problem in this economy. (worth 2 points)

The resource constraint of the economy can be written as:

$$c_t + k_{t+1} + h_{t+1} = Bk_t^{\alpha}h_t^{1-\alpha} + (1-\delta)k_t + (1-\delta)h_t$$

The social planner's problem can be written as:

$$\max_{c,k,h} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t. } c_t + k_{t+1} + h_{t+1} = Bk_t^{\alpha} h_t^{1-\alpha} + (1-\delta)k_t + (1-\delta)h_t \;\; \forall t$$

The FOCs are as follows:

$$c_t: \; \beta^t u'(c_t) = \mu_t$$
$$k_{t+1}: \; \mu_t = \mu_{t+1}\left(1 + \alpha B\left(\frac{k_{t+1}}{h_{t+1}}\right)^{\alpha-1} - \delta\right)$$
$$h_{t+1}: \; \mu_t = \mu_{t+1}\left(1 + (1-\alpha) B\left(\frac{k_{t+1}}{h_{t+1}}\right)^{\alpha} - \delta\right)$$

Combining the last two FOCs, we have: $\frac{k_{t+1}}{h_{t+1}} = \frac{\alpha}{1-\alpha} \quad \forall t$

1. **What is the optimal k/h ratio?**

   Our optimal k/h ratio is $\alpha/(1-\alpha)$

2. **What is the equilibrium growth rate of consumption?**

   Using our FOC for c_t and k_{t+1}, we have:
   $$\frac{c_{t+1}}{c_t} = \left[\beta\left(1 + \alpha^{\alpha}(1-\alpha)^{1-\alpha} B - \delta\right)\right]^{\theta}$$

(c) Solve for the decentralized competitive equilibrium: Assume that labor supply is exogenous, and that the borrowing constraint is never binding. (worth 2 points)

1. **Prove that the household budget constraint can be written as:**
   $c_t + k_{t+1} + h_{t+1} + b_{t+1} = (1 + w_t - \delta)h_t + (1 + r_t - \delta)k_t + (1 + R_t)b_t$

   The household budget constraint is:
   $c_t + i_t^k + i_t^h + b_{t+1} = y_t = w_t h_t + r_t k_t + (1 + R_t)b_t$

   The accumulation equations are
   $k_{t+1} = (1-\delta)k_t + i_t^k$
   $h_{t+1} = (1-\delta)h_t + i_t^h$

   Combining them, we have our final budget constraint:
   $c_t + k_{t+1} + h_{t+1} + b_{t+1} = (1 + w_t - \delta)h_t + (1 + r_t - \delta)k_t + (1 + R_t)b_t$

2. **What is the arbitrage condition of the economy?**

   It must be the case that the individual is indifferent between investing in bonds, physical capital and human capital. Thus, we have the following condition:

$$(1+w_t-\delta)=(1+r_t-\delta)=(1+R_t)$$
$$(w_t-\delta)=(r_t-\delta)=R_t$$

Thus, it must also be the case that $w_t=r_t$ in equilibrium.

3. **Use the arbitrage condition to show that the household's budget constraint can be written as: $c_t+a_{t+1}=(1+R_t)a_t$, where $a_t=b_t+k_t+h_t$, and use the FOCs to find the Euler Condition.**

   From Part 2, it is straightforward to see that the arbitrage condition implies that the optimal allocation of savings is undetermined for any household thus giving us the new budget constraint above. From this, we can easily derive our usual Euler condition.

$$\frac{c_{t+1}}{c_t}=\left[\beta(1+R_t)\right]^{\theta}$$

4. **Using the firms profit maximizing behavior, what are the equilibrium interest rate and wage rate (for human capital)?**

   The firm has the following maximization problem:

$$\max_{K,L} BK_t^{\alpha}(h_tL_t)^{1-\alpha}-w_th_tL_t-r_tK_t$$

   The FOCS are given by:

$$r_t=\alpha Bk_t^{\alpha-1}h_t^{1-\alpha}$$
$$w_t=(1-\alpha)Bk_t^{\alpha}h_t^{-\alpha}$$

5. **What is the optimal k/h ratio?**

   From the arbitrage condition, we know that $w_t=r_t$, thus using the firms FOCs, we have the optimal k/h ratio is $\alpha/(1-\alpha)$.

6. **What is the growth rate of consumption?**

   Using the arbitrage condition, and plugging in for R_t in the individual's Euler Condition, we have:

$$\frac{c_{t+1}}{c_t}=\left[\beta\left(1+\alpha^{\alpha}(1-\alpha)^{1-\alpha}B-\delta\right)\right]^{\theta}$$

(d) Does the consumption growth in the competitive equilibrium from Part C match the social planner's optimal consumption growth rate you found in Part B? Why are they different or the same? How is this model different than the learning by doing model in Question #1? (worth 1 point)

Yes, the decentralized equilibrium matches the equilibrium we found in the social planner's problem. The two growth rates are the same because there are no externalities in this model (i.e. the private return to each capital equals the social return). Individual's explicitly take account of the return to investing in their human capital in this model whereas in question #1, human capital was produced as an externality from the use of physical capital and the private return to physical capital was less than the social return.

(e) Assume the government decides to tax a firm's output in this economy. In particular, assume the government takes a fraction τ^y of all firms' output. The government then balances its budget by returning the tax as a lump sum payment T_t to individuals. Resolve for the decentralized equilibrium as done in Part C. (worth 2 points)

Nothing of the individual's optimization problem will be changed because of the above tax system. The lump sum tax will show up in the budget constraint, but it will now affect the FOCs. Thus, we have the same Euler condition as before. However, the firm's optimization problem will be changed. The firm's profit maximizing FOCs will now be given as:

The firm has the following maximization problem:

$$\max_{K,L}\left(1-\tau^y\right)BK_t^{\alpha}\left(h_t L_t\right)^{1-\alpha}-w_t h_t L_t - r_t K_t$$

The FOCS are given by:

$$r_t=\left(1-\tau^y\right)\alpha Bk_t^{\alpha-1}h_t^{1-\alpha}$$
$$w_t=\left(1-\tau^y\right)\left(1-\alpha\right)Bk_t^{\alpha}h_t^{-\alpha}$$

1. Is the optimal ratio of k/h affected? Why or why not?

No, using $r_t=w_t$, we immediately see that the output tax does not affect the optimal ratio of k/h. This is because the output tax equally distorts the return to each type of capital.

2. How does the equilibrium return to physical capital change?

While the optimal ratio of k/h is unchanged, the equilibrium return to capital is now only $r_t=\left(1-\tau^y\right)\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}B$. The tax reduces the return to capital.

3. How does the tax affect the growth of consumption?

Plugging in for $R_t=r_t-\delta$ into our Euler condition, we see that the lower return to capital results in lower growth rate of consumption:

$$\frac{c_{t+1}}{c_t}=\left[\beta\left(1+\left(1-\tau^y\right)\alpha^\alpha\left(1-\alpha\right)^{1-\alpha}B-\delta\right)\right]^\theta$$

(f) Now assume that the government taxes the private return of human capital for individuals. In other words, the private return is now given by $\left(1-\tau^h\right)w_t$ where $\tau^h>0$. The government returns the tax as a lump sum subsidy to individuals. Resolve for the decentralized equilibrium. **(worth 2 points)**

The individual's Euler condition will remain unchanged in this problem since the lump sum tax has no effect on the FOCs, but the arbitrage condition between physical and human capital now implies that $\left(1-\tau^w\right)w_t=r_t=R_t+\delta$.

The firm's FOCs will be the same as in Part D.

$$r_t=\alpha Bk_t^{\alpha-1}h_t^{1-\alpha}$$
$$w_t=\left(1-\alpha\right)Bk_t^\alpha h_t^{-\alpha}$$

1. **Is the optimal ratio of k/h affected now? Why or why not?**

   Combining the firm's FOCs with the new arbitrage condition, we find that the optimal k/h is given by: $k/h=\frac{\alpha}{\left(1-\tau^w\right)\left(1-\alpha\right)}$. Thus, for $\tau^w>0$, we see that the optimal ratio is now higher. i.e. There will be more physical capital per unit of human capital in the new equilibrium. This is just as we might expect… the government tax on human capital results in a greater weight placed on physical capital in the new equilibrium.

2. **How does the tax affect the equilibrium prices for capital and the growth of consumption?**

   Plugging in for the new optimal ratio, we see that the new equilibrium return to physical and human capital is:

   $$r_t=\left(1-\tau^w\right)^{1-\alpha}\alpha^\alpha\left(1-\alpha\right)^{1-\alpha}B$$
   $$w_t=\left(1-\tau^w\right)^{-\alpha}\alpha^\alpha\left(1-\alpha\right)^{1-\alpha}B$$

   The equilibrium return to physical capital is lower (because there will be less human capital per unit of physical capital), but the equilibrium return to human capital (before the subsidy) will be higher because it is used with larger quantities of physical capital.

   Plugging in for $R_t=r_t-\delta$ in the Euler condition, we have:

   $$\frac{c_{t+1}}{c_t}=\left[\beta\left(1+\left(1-\tau^w\right)^{1-\alpha}\alpha^\alpha\left(1-\alpha\right)^{1-\alpha}B-\delta\right)\right]^\theta$$

   By lowering the return to capital, the tax reduces the growth rate of consumption.