Solutions

14.06 Problem Set 1 2004 - Solow Model

Prof: Marios Angeletos

Question 1: Begin by normalizing all variables per unit of effective labor so that

$$k_t = \frac{K_t}{A_t L_t}, y_t = \frac{Y_t}{A_t L_t}, i_t = \frac{I_t}{A_t L_t}$$

This allows us to write the production function as

$$y_t = k_t^\alpha.$$

The evolution of capital is (now using the same timing as in the notes):

$$\begin{aligned}
K_{t+1} &= (1-\delta) K_t + I_t \\
\frac{K_{t+1}}{A_t L_t} &= (1-\delta) \frac{K_t}{A_t L_t} + \frac{I_t}{A_t L_t} \\
\frac{K_{t+1}(1+n)(1+g)}{A_{t+1} L_{t+1}} &= (1-\delta) k_t + i_t \\
k_{t+1}(1+n)(1+g) &= (1-\delta) k_t + i_t
\end{aligned}$$

For $k_{t+1} \approx k_t$ and $ng \approx 0$ (ie n and g are small) we can approximate this by

$$k_{t+1} \approx (1-\delta-n-g) k_t + i_t$$

Now by Solow's assumption:

$$i_t = s y_t = s k_t^\alpha$$

so that the evolution of capital is governed by

$$k_{t+1} \approx (1-\delta-n-g) k_t + s k_t^\alpha.$$

The steady state level of capital is thus

$$\begin{aligned}
k^* &= (1-\delta-n-g) k^* + s (k^*)^\alpha \\
k^* &= \left(\frac{s}{\delta+n+g}\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

The steady state level of output (per effective worker) is thus

$$y^* = \left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}$$

and for consumption

$$c^* = (1-s)\left(\frac{s}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}.$$

Steady state values of C, Y, I, K grow at a rate $n+g$ so that in the steady state $C_t = (1+n)(1+g) C_{t+1}$.

Finally, output per worker is

$$\frac{Y_t}{L_t} = y_t A_t$$

Since $y_t = y^*$ in the steady state then $\frac{Y_t}{L_t}$ must grow at the same rate as technology (g) so that

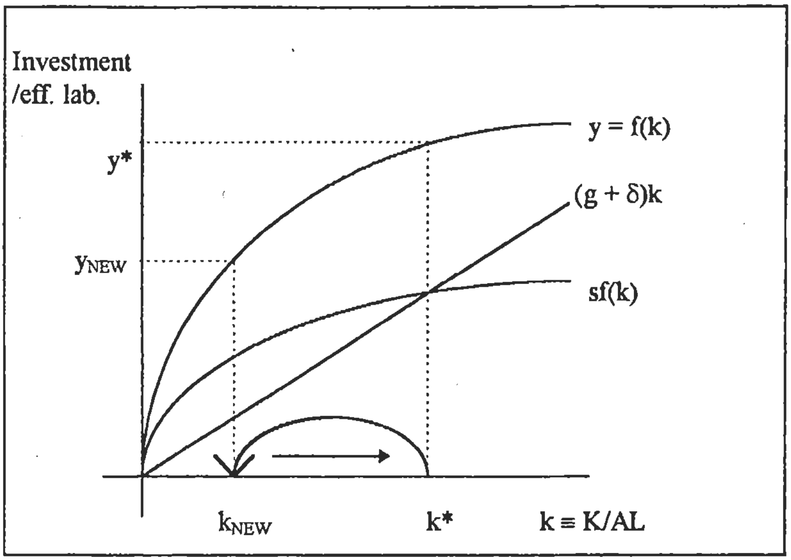
$$\frac{Y_{t+1}}{L_{t+1}} = (1+g)\frac{Y_t}{L_t}.$$

Similarly consumption and investment per worker will grow at the same rate.

Problem 1.4

(a) At some time, call it t_0, there is a discrete upward jump in the number of workers. This reduces the amount of capital per unit of effective labor from k^* to k_{NEW}. We can see this by simply looking at the definition, $k \equiv K/AL$. An increase in L without a jump in K or A causes k to fall. Since $f'(k) > 0$, this fall in the amount of capital per unit of effective labor reduces the amount of output per unit of effective labor as well. In the figure below, y falls from y^* to y_{NEW}.

(b) Now at this lower k_{NEW}, actual investment per unit of effective labor exceeds break-even investment per unit of effective labor. That is, $sf(k_{NEW}) > (g + \delta)k_{NEW}$. The economy is now saving and investing more than enough to offset depreciation and technological progress at this lower k_{NEW}. Thus k begins rising back toward k^*. As capital per unit of effective labor begins rising, so does output per unit of effective labor. That is, y begins rising from y_{NEW} back toward y^*.

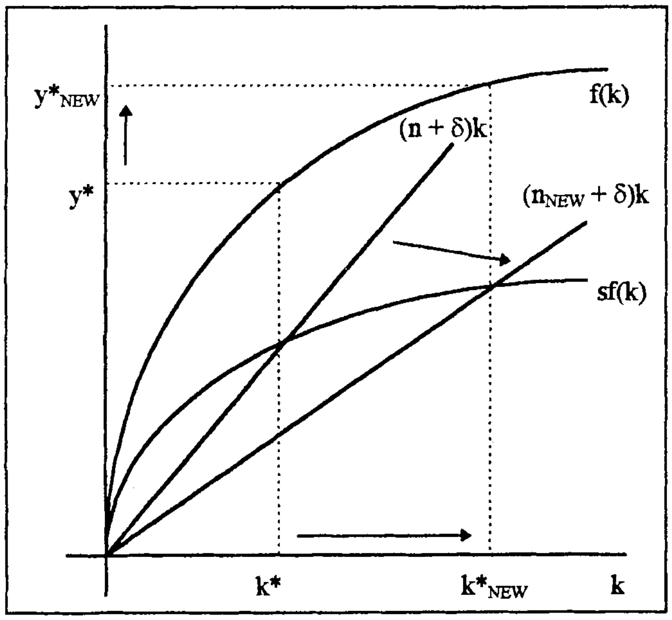


(c) Capital per unit of effective labor will continue to rise until it eventually returns to the original level of k^*. At k^*, investment per unit of effective labor is again just enough to offset technological progress and depreciation and keep k constant. Since k returns to its original value of k^* once the economy again returns to a balanced growth path, output per unit of effective labor also returns to its original value of $y^* = f(k^*)$.

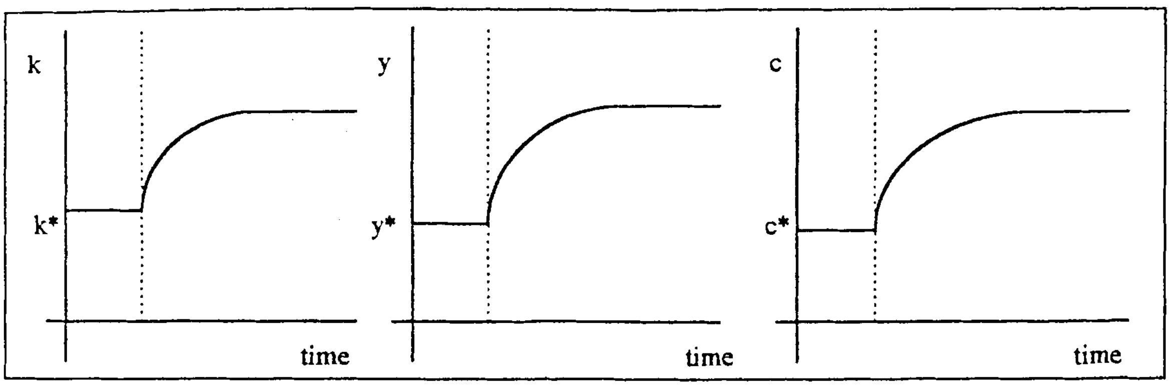
Problem 1.6

(a) Since there is no technological progress, we can carry out the entire analysis in terms of capital and output per worker rather than capital and output per unit of effective labor. With A constant, they behave the same. Thus we can define $y \equiv Y/L$ and $k \equiv K/L$.

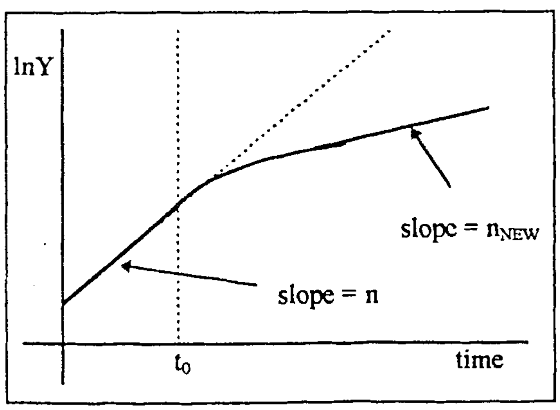
The fall in the population growth rate makes the break-even investment line flatter. In the absence of technological progress, the per unit time change in k, capital per worker, is given by $\dot{k} = sf(k) - (\delta + n)k$. Since $\dot{k}$ was 0 before the decrease in n -- the economy was on a balanced growth path -- the decrease in n causes $\dot{k}$ to become positive. At k*, actual investment per worker, sf(k*), now exceeds break-even investment per worker, $(n_{NEW} + \delta)k^*$. Thus k moves to a new higher balanced growth path level. See the figure at right.



As k rises, y -- output per worker -- also rises. Since a constant fraction of output is saved, c -- consumption per worker -- rises as y rises. This is summarized in the figures below.



(b) By definition, output can be written as $Y \equiv Ly$. Thus the growth rate of output is $\dot{Y}/Y = \dot{L}/L + \dot{y}/y$. On the initial balanced growth path, $\dot{y}/y = 0$ -- output per worker is constant -- so $\dot{Y}/Y = \dot{L}/L = n$. On the final balanced growth path, $\dot{y}/y = 0$ again -- output per worker is constant again -- and so $\dot{Y}/Y = \dot{L}/L = n_{NEW} < n$. In the end, output will be growing at a permanently lower rate.



What happens during the transition? Examine the production function $Y = F(K,AL)$. On the initial balanced growth path AL, K and thus Y are all growing at rate n. Then suddenly AL begins growing at some new lower rate n_{NEW}. Thus suddenly Y will be growing at some rate between that of K (which is growing at n) and that of AL (which is growing at n_{NEW}). Thus, during the transition, output grows more rapidly than it will on the new balanced growth path, but less rapidly than it would have without the decrease in population growth. As output growth gradually slows down during the transition, so does capital growth until finally K, AL, and Y are all growing at the new lower n_{NEW}.

Problem 1.9

(a) Define the marginal product of labor as $w \equiv \partial F(K,AL)/\partial L$. Then write the production function as $Y = ALf(k) = ALf(K/AL)$. Taking the partial derivative of output with respect to L yields

(1) $w \equiv \partial Y/\partial L = ALf'(k)[-K/AL^2] + Af(k) = A[(-K/AL)f'(k) + f(k)] = A[f(k) - kf'(k)]$,

as required.

(b) Define the marginal product of capital as $r \equiv [\partial F(K,AL)/\partial K] - \delta$. Again, writing the production function as $Y = ALf(k) = ALf(K/AL)$ and now taking the partial derivative of output with respect to K yields

(2) $r \equiv [\partial Y/\partial K] - \delta = ALf'(k)[1/AL] - \delta = f'(k) - \delta$.

Substitute equations (1) and (2) into $wL + rK$:

$$wL + rK = A[f(k) - kf'(k)]\,L + [f'(k) - \delta]K = ALf(k) - f'(k)[K/AL]AL + f'(k)K - \delta K.$$

Simplifying gives us

(3) $wL + rK = ALf(k) - f'(k)K + f'(k)K - \delta K = Alf(k) - \delta K \equiv ALF(K/AL,\ 1) - \delta K$.

Finally, since F is constant returns to scale, equation (3) can be rewritten as

(4) $wL + rK = F(ALK/AL,\ AL) - \delta K = F(K,\ AL) - \delta K$.

(c) As shown above, $r = f'(k) - \delta$. Since δ is a constant and since k is constant on a balanced growth path, so is $f'(k)$ and thus so is r. In other words, on a balanced growth path, $\dot{r}/r = 0$. Thus the Solow model does exhibit the property that the return to capital is constant over time.

Since capital is paid its marginal product, the share of output going to capital is rK/Y. On a balanced growth path,

(5) $$\frac{\dot{(rK/Y)}}{(rK/Y)} = \dot{r}/r + \dot{K}/K - \dot{Y}/Y = 0 + (n+g) - (n+g) = 0.$$

Thus, on a balanced growth path, the share of output going to capital is constant. Since the shares of output going to capital and labor sum to one, this implies that the share of output going to labor is also constant on the balanced growth path.

We need to determine the growth rate of the marginal product of labor, w, on a balanced growth path. As shown above, $w = A[f(k) - kf'(k)]$. Taking the time derivative of the log of this expression yields the growth rate of the marginal product of labor:

(6) $$\frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \frac{\dot{[f(k) - kf'(k)]}}{[f(k) - kf'(k)]} = g + \frac{[f'(k)\dot{k} - \dot{k}f'(k) - kf''(k)\dot{k}]}{f(k) - kf'(k)} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

On a balanced growth path $\dot{k} = 0$ and so $\dot{w}/w = g$. That is, on a balanced growth path, the marginal product of labor rises at the rate of growth of the effectiveness of labor.

(d) As shown in part (c), the growth rate of the marginal product of labor is

(6) $$\frac{\dot{w}}{w} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

If $k < k^*$, then as k moves toward k^*, $\dot{w}/w > g$. This is true because the denominator of the second term on the right-hand side of equation (6) is positive because f(k) is a concave function. The numerator of that same term is positive because k and $\dot{k}$ are positive and $f''(k)$ is negative. Thus, as k rises toward k^*, the marginal product of labor grows faster than on the balanced growth path. Intuitively, the marginal product of labor rises by the rate of growth of the effectiveness of labor on the balanced growth path. As we move from k to k^*, however, the amount of capital per unit of effective labor is also rising which also makes labor more productive and this increases the marginal product of labor even more.

The growth rate of the marginal product of capital, r, is

(7) $$\frac{\dot{r}}{r} = \frac{\dot{[f'(k)]}}{f'(k)} = \frac{f''(k)\dot{k}}{f'(k)}.$$

As k rises toward k^*, this growth rate is negative since $f'(k) > 0$, $f''(k) < 0$ and $\dot{k} > 0$. Thus, as the economy moves from k to k^*, the marginal product of capital falls. That is, it grows at a rate less than on the balanced growth path where its growth rate is 0.