Chapter 15

Static Applications with Incomplete Information

This chapter is devoted to economic applications with incomplete information. They are meant to illustrate the common techniques in computing Bayesian Nash equilibria in static games of incomplete information. There are four applications. The first application is Cournot duopoly, where I illustrate how to computes the Bayesian Nash equilibria when there is a continuum of actions but finitely many types. The next two applications are the first-price auction and double auction. In these applications, there are a continuum of actions and a continuum of types. In that case, it is not easy to compute all equilibria, and one often considers equilibria with certain functional forms. Here, the focus will be on (i) symmetric, linear equilibrium, (ii) symmetric but not necessarily linear equilibrium, and (iii) linear but not necessarily symmetric equilibrium. I will explain what symmetry and linearity means when we come there. Finally, I will consider coordination games with incomplete information. With complete information, these games often has multiple equilibria. When there is enough incomplete information, multiple equilibria disappears. I will illustrate this using "monotone" equilibria, in which there is a cutoff value such that players play one action below the cutoff and another action above the cutoff. My technical objective in the last example is to illustrate how to compute monotone equilibria (when they exist).

15.1 Cournot Duopoly with incomplete information

Consider a Cournot duopoly with inverse-demand function

$$P(Q) = a - Q$$

where $Q = q_1 + q_2$. The marginal cost of Firm 1 is $c = 0$, and this is common knowledge. Firm 2's marginal cost c_2 is its own private information. It can take values of

$$\begin{array}{ll} c_H & \text{with probability } \theta \text{, and} \\ c_L & \text{with probability } 1-\theta. \end{array}$$

Each firm maximizes its expected profit.

Here, Firm 1 has just one type, and Firm 2 has two types: c_H and c_L. Hence, a strategy of Firm 1 is a real number q_1, while a strategy of Firm 2 is two real numbers $q_2(c_H)$ and $q_2(c_L)$, one for when the cost is c_H and one for when the cost is c_L.

Bayesian Nash Equilibrium A Bayesian Nash equilibrium is a triplet $(q_1^*, q_2^*(c_H), q_2^*(c_L))$ of real numbers, where q_1^* is the production level of Firm 1, $q_2^*(c_H)$ is the production level of type c_H of Firm 2, and $q_2^*(cL)$ is the production level of type c_L of Firm 2. In equilibrium each type plays a best response. First consider the high-cost type c_H of Firm 2. In equilibrium, that type knows that Firm 1 produces q_1^*. Hence, its production level, $q_2^*(c_H)$, solves the maximization problem

$$\max_{q_2}(P - c_H)q_2 = \max_{q_2}\left[a - q_1^* - q_2 - c_H\right]q_2.$$

Hence,

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \tag{15.1}$$

Now consider the low-cost type c_L of Firm 2. In equilibrium, that type also knows that Firm 1 produces q_1^*. Hence, its production level, $q_2^*(c_L)$, solves the maximization problem

$$\max_{q_2}\left[a - q_1^* - q_2 - c_L\right]q_2.$$

Hence,

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}. \tag{15.2}$$

The important point here is that both types consider the same q_1^*, as that is the strategy of Firm 1, whose type is known by both types of Firm 2. Now consider Firm 1. It has one type. Firm 1 knows the strategy of Firm 2, but since it does not know which type of Firm 2 it faces, it does not know the production level of Firm 2. In Firm 1's view, the production level of Firm 2 is $q_2^*(c_H)$ with probability θ and $q_2^*(c_L)$ with probability $1-\theta$. Hence, the expected profit of Firm 1 from production level q_1 is

$$\begin{aligned} U_1(q_1, q_2^*) &= \theta\left[a - q_1 - q_2^*(c_H)\right] q_1 + (1-\theta)\left[a - q_1 - q_2^*(c_L)\right] q_1 \\ &= \left\{a - q_1 - \left[\theta q_2^*(c_H) + (1-\theta) q_2^*(c_L)\right]\right\} q_1. \end{aligned}$$

The equality is due to the fact that the production level q_2 of Firm 2 enters the payoff $[a - q_1 - q_2]\, q_1 = [a - q_1]\, q_1 - q_1 q_2$ of Firm 1 linearly. The term

$$E[q_2] = \theta q_2^*(c_H) + (1-\theta) q_2^*(c_L)$$

is the expected production level of Firm 2. Hence, the expected profit of Firm 1 just his profit from expected production level:

$$U(q_1, q_2^*) = (a - q_1 - E[q_2])\, q_1.$$

Its strategy q_1^* solves the maximization problem

$$\max_{q_1} U(q_1, q_2^*).$$

In this particular case, it is a best response to the expected production level:

$$q_1^* = \frac{a - E[q_2]}{2} = \frac{a - \left[\theta q_2^*(c_H) + (1-\theta) q_2^*(c_L)\right]}{2}. \tag{15.3}$$

It is important to note that the equilibrium action is a best response to expected strategy of the other player when *and only when* the action of the other players affect the payoff of the player linearly, as in this case.[1] In particular, when the other players' actions have a non-linear effect on the payoff of a player, his action may **not** be a best response to expected action of the others. It is a common mistake to take a player's action as a best response to the expected action of others; you must avoid it.

To compute the Bayesian Nash equilibrium, one simply needs to solve the three linear equations (15.1), (15.2), and (15.3) for $q_1^*, q_2^*(c_L), q_2^*(c_H)$. Write

[1] To be more precise, when $\partial U_i / \partial q_i$ is linear in q_j.

$$\begin{pmatrix} q_1^* \\ q_2^*(c_H) \\ q_2^*(c_L) \end{pmatrix} = \begin{bmatrix} 2 & \theta & 1-\theta \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} \begin{pmatrix} a \\ a - c_H \\ a - c_L \end{pmatrix},$$

yielding

$$\begin{aligned} q_2^*(c_H) &= \frac{a - 2c_H}{3} + \frac{(1-\theta)(c_H - c_L)}{6} \\ q_2^*(c_L) &= \frac{a - 2c_L}{3} - \frac{\theta(c_H - c_L)}{6} \\ q_1^* &= \frac{a + \theta c_H + (1-\theta)c_L}{3}. \end{aligned}$$

15.2 First-price Auction

There is an object to be sold. Two bidders want to buy it through an auction. Simultaneously, each bidder i submits a bid $b_i \geq 0$. Then, the highest bidder wins the object and pays her bid. If they bid the same number, then the winner is determined by a coin toss. The value of the object for bidder i is v_i, which is privately known by bidder i. That is, v_i is the type of bidder i. Assume that v_1 and v_2 are "independently and identically distributed" with uniform distribution over $[0, 1]$. This precisely means that knowing her own value v_i, bidder i believes that the other bidder's value v_j is distributed with uniform distribution over $[0, 1]$, and the type space of each player is $[0, 1]$. Recall that the beliefs of a player about the other player's types may depend on the player's own type. Independence assumes that it doesn't.

Formally, the Bayesian game is as follows. Actions are b_i, coming from the action spaces $[0, \infty)$; types are v_i, coming from the type spaces $[0, 1]$; beliefs are uniform distributions over $[0, 1]$ for each type, and the utility functions are given by

$$u_i(b_1, b_2, v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j, \\ \frac{v_i - b_i}{2} & \text{if } b_i = b_j, \\ 0 & \text{if } b_i < b_j. \end{cases}$$

In a Bayesian Nash equilibrium, each type v_i maximizes the expected payoff

$$E\left[u_i(b_1, b_2, v_1, v_2)|v_i\right] = (v_i - b_i)\Pr\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)\Pr\{b_i = b_j(v_j)\} \quad (15.4)$$

over b_i.

Next, we will compute the Bayesian Nash equilibria. First, we consider a special equilibrium. The technique we will use here is a common technique in computing Bayesian Nash equilibria, and pay close attentions to the steps.

15.2.1 Symmetric, linear equilibrium

This section is devoted to the computation of a symmetric, linear equilibrium. Symmetric means that equilibrium action $b_i(v_i)$ of each type v_i is given by

$$b_i(v_i) = b(v_i)$$

for some function b from type space to action space, where b is the same function for all players. Linear means that b is an affine function of v_i, i.e.,

$$b_i(v_i) = a + cv_i.$$

To compute symmetric, linear equilibrium, one follows the following steps.

Step 1 *Assume a symmetric linear equilibrium:*

$$\begin{aligned} b_1^*(v_1) &= a + cv_1 \\ b_2^*(v_2) &= a + cv_2 \end{aligned}$$

for all types v_1 and v_2 for some constants a and c, that will be determined later. The important thing here is the constants do not depend on the players or their types.

Step 2 *Compute the best reply function of each type.* Fix some type v_i. To compute her best reply, first note that $c > 0$.[2] Then, for any fixed value b_i,

$$\Pr\{b_i = b_j^*(v_j)\} = 0, \tag{15.5}$$

as b_j is strictly increasing in v_j by Step 1. It is also true that $a \leq b_i(v_i) \leq v_i$. [You need to figure this out!] Hence,

[2] If $c = 0$, both bidders bid a independent of their type. Then, bidding 0 is a better response for a type $v_i < a$; a type $v_i > a$ also has an incentive to deviate by increasing her bid slightly.

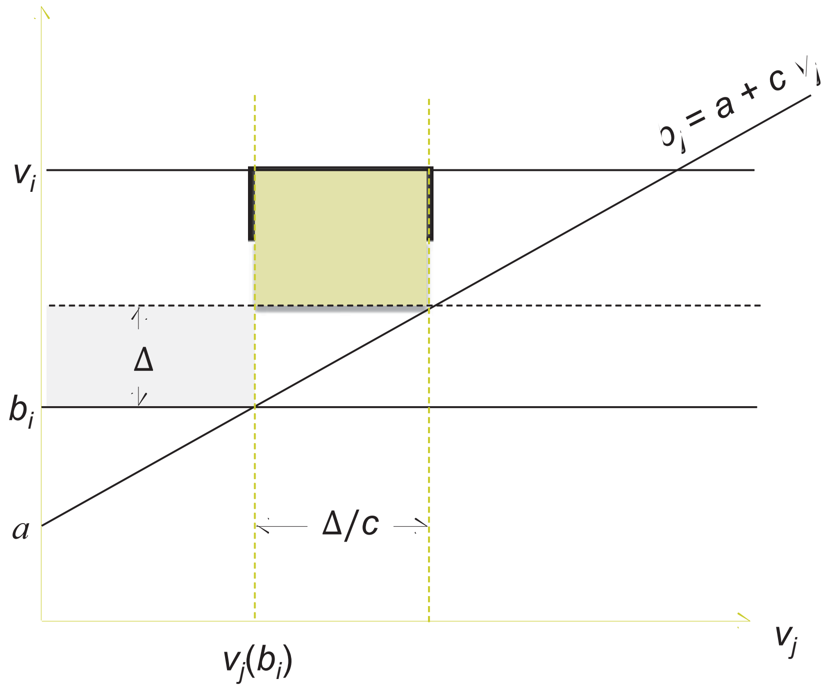

Figure 15.1: Payoff as a function of bid in first-price auction

$$
\begin{aligned}
E\left[u_i(b_i, b_j^*, v_1, v_2)|v_i\right] &= (v_i - b_i)\Pr\{b_i \geq a + cv_j\} \\
&= (v_i - b_i)\Pr\{v_j \leq \frac{b_i - a}{c}\} \\
&= (v_i - b_i) \cdot \frac{b_i - a}{c}.
\end{aligned}
$$

Here, the first equality is obtained simply by substituting (15.5) to (15.4). The second equality is simple algebra, and the third equality is due to the fact that v_j is distributed by uniform distribution on $[0, 1]$. [If you are taking this course, the last step must be obvious to you!]

For a graphical derivation, consider Figure 15.1. The payoff of i is $v_i - b_i$ when $v_j \leq v_j(b_i) = (b_i - a)/c$ and is zero otherwise. Hence, his expected payoff is the integral[3]

$$\int_0^{v_j(b_i)} (v_i - b_i)\, dv_j.$$

This is the area of the rectangle that is between 0 and $v_j(b)$ horizontally and between b_i and v_i vertically: $(v_i - b_i)\, v_j(b_i)$.

To compute the best reply, we must maximize the last expression over b_i. Taking the derivative and setting equal to zero yields

$$b_i = \frac{v_i + a}{2}. \tag{15.6}$$

Graphically, as plotted in Figure 15.1, when b_i is increased by an amount of Δ, $v_j(b_i)$ increases by an amount of Δ/c. To the expected payoff, this adds a rectangle of size $(v_i - b_i - \Delta)\,\Delta/c$, which is approximately $(v_i - b_i)\,\Delta/c$ when Δ is small, and substact a rectangle of size $v_j(b_i)\,\Delta$. At the optimum these two must be equal:

$$(v_i - b_i)\,\Delta/c = v_j(b_i)\,\Delta,$$

yielding (15.6) above.

Remark 15.1 *Note that we took an integral to compute the expected payoff and took a derivative to compute the best response. Since the derivative is an inverse of integral, this involves unnecessary calculations in general. In this particular example, the calculations were simple. In general those unnecessary calculations may be the hardest step. Hence, it is advisable that one leaves the integral as is and use Leibnitz rule[4] to differentiate it to obtain the first-order condition. Indeed, the graphical derivation above does this.*

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[3] If v_j were not uniformly distributed on $[0,1]$, then it would have been the integral

$$\int_0^{v_j(b_i)} (v_i - b_i)\, f(v_j)\, dv_j = (v_i - b_i)\, F(v_j(b_i))$$

where f and F is the probability density and cumulative distribution functions of v_j, respectively.

[4] Leibnitz Rule:

$$\frac{\partial}{\partial x}\int_{t=L(x,y)}^{U(x,y)} f(x,y,t)\, dt = \frac{\partial U}{\partial x}\cdot f(x,y,U(x,y)) - \frac{\partial L}{\partial x}\cdot f(x,y,L(x,y)) + \int_{t=L(x,y)}^{U(x,y)} \frac{\partial}{\partial x} f(x,y,t)\, dt.$$

Step 3 *Verify that best -reply functions are indeed affine*, i.e., b_i is of the form $b_i = a + cv_i$. Indeed, we rewrite (15.6) as

$$b_i = \frac{1}{2}v_i + \frac{a}{2}. \tag{15.7}$$

Check that both $1/2$ and $a/2$ are constant, i.e., they do not depend on v_i, and they are same for both players.

Step 4 *Compute the constants a and c.* To do this, observe that in order to have an equilibrium, the best reply b_i in (15.6) must be equal to $b_i^*(v_i)$ for each v_i. That is,

$$\frac{1}{2}v_i + \frac{a}{2} = cv_i + a.$$

must be an identity, i.e. it must remain true for all values of v_i. Hence, the coefficient of v_i must be equal in both sides:

$$c = \frac{1}{2}.$$

The intercept must be same in both sides, too:

$$a = \frac{a}{2}.$$

Thus,

$$a = 0.$$

This yields the symmetric, linear Bayesian Nash equilibrium:

$$b_i(v_i) = \frac{1}{2}v_i.$$

15.2.2 Any symmetric equilibrium

I now compute a symmetric Bayesian Nash equilibrium without assuming that b is linear. Assume that b is strictly increasing and differentiable.

Step 1 *Assume a Bayesian Nash equilibrium of the form*

$$\begin{aligned} b_1^*(v_1) &= b(v_1) \\ b_2^*(v_2) &= b(v_2) \end{aligned}$$

for some increasing, differentiable function b.

Step 2 *Compute the best reply of each type, or compute the first-order condition that must be satisfied by the best reply.* To this end, compute that, given that the other player j is playing according to equilibrium, the expected payoff of playing b_i for type v_i is

$$\begin{aligned} E\ \ u_i\left(b_i, b_j^*, v_1, v_2\right)|v_i &= (v_i - b_i)\Pr\{b_i \geq b(v_j)\} \\ &= (v_i - b_i)\Pr\{v_j \leq b^{-1}(b_i)\} \\ &= (v_i - b_i)b^{-1}(b_i), \end{aligned}$$

where b^{-1} is the inverse of b. Here, the first equality holds because b is strictly increasing; the second equality is obtained by again using the fact that b is increasing, and the last equality is by the fact that v_j is uniformly distributed on $[0,1]$. The first-order condition is obtained by taking the partial derivative of the last expression with respect to b_i and setting it equal to zero. Then, the first-order condition is

$$-b^{-1}\left(b_i^*\left(v_i\right)\right) + \left(v_i - b_i^*\right) \left.\frac{db^{-1}}{db_i}\right|_{b_i = b_i^*(v_i)} = 0.$$

Using the formula on the derivative of the inverse function, this can be written as

$$-b^{-1}\left(b_i^*\left(v_i\right)\right) + \left(v_i - b_i^*\left(v_i\right)\right) \left.\frac{1}{b'(v)}\right|_{b(v) = b_i^*(v)} = 0. \tag{15.8}$$

Step 3 *Identify the best reply with the equilibrium action*, towards computing the equilibrium action. That is, set

$$b_i^*\left(v_i\right) = b\left(v_i\right).$$

Substituting this in (15.8), obtain

$$-v_i + \left(v_i - b\left(v_i\right)\right)\frac{1}{b'\left(v_i\right)} = 0. \tag{15.9}$$

Most of the time the differential equation does not have a closed-form solution. In that case, one suffices with analyzing the differential equation. Luckily, in this case the differential equation can be solved, easily. By simple algebra, we rewrite the differential equation as

$$b'\left(v_i\right)v_i + b\left(v_i\right) = v_i.$$

Hence,

$$\frac{d\left[b\left(v_i\right)v_i\right]}{dv_i} = v_i.$$

Therefore,

$$b\left(v_i\right)v_i = v_i^2/2 + const,$$

for some constant $const$. Since the equality also holds $v_i = 0$, it must be that $const = 0$. Therefore,

$$b\left(v_i\right) = v_i/2.$$

In this case, we were lucky. In general, one obtains a differential equation as in (15.9), but the equation is not easily solvable in general. Make sure that you understand the steps until finding the differential equation well.

15.2.3 General Case

I have so far assumed that the types are uniformly distributed. Assume now instead that the types are independently and identically distributed with a probability density function f and cumulative distribution function F. (In the case of uniform, f is 1 and F is identity on $[0,1]$.) To compute the symmetric equilibria in increasing differentiable strategies, observe that the expected payoff in Step 2 is

$$E \;\; u_i \;\; b_i, b_j^*, v_1, v_2 \;\; |v_i \;\; = \left(v_i - b_i\right)\Pr\{v_j \leq b^{-1}\left(b_i\right)\} = (v_i - b_i)F \;\; b^{-1}\left(b_i\right) \;\; .$$

The first-order condition for best reply is then

$$-F \;\; b^{-1}\left(b_i^*\left(v_i\right)\right) \;\; + \left(v_i - b_i^*\left(v_i\right)\right)f \;\; b^{-1}\left(b_i^*\left(v_i\right)\right) \;\; \frac{db^{-1}}{db_i} \;\; {}_{b_i = b_i^*(v_i)} \;\; = 0.$$

Using the formula on the derivative of the inverse function, this can be written as

$$-F \;\; b^{-1}\left(b_i^*\left(v_i\right)\right) \;\; + \left(v_i - b_i^*\left(v_i\right)\right)f \;\; b^{-1}\left(b_i^*\left(v_i\right)\right) \;\; \frac{1}{b\ (v)} \;\; {}_{b(v) = b_i^*(v)} \;\; = 0.$$

In Step 3, towards i*dentifying the best reply with the equilibrium action*, one substutes the equality $b_i^*\left(v_i\right) = b\left(v_i\right)$ in this equation and obtains

$$-F\left(v_i\right) + \left(v_i - b\left(v_i\right)\right)f\left(v_i\right)\frac{1}{b\ \left(v_i\right)} = 0.$$

Arranging the terms, one can write this as a usual differential equation:

$$b\ (v_i) F(v_i) + b(v_i) f(v_i) = v_i f(v_i).$$

The same trick in the case of uniform distribution applies more generally. One can write the above differential equation as

$$\frac{d}{dv_i}[b(v_i) F(v_i)] = v_i f(v_i).$$

By integrating both sides, one then obtains the solution

$$b(v_i) = \frac{\int_0^{v_i} v f(v)\, dv}{F(v_i)}.$$

One can further simplify this solution by integrating the right hand side by parts:

$$b(v_i) = \frac{v_i F(v_i) - \int_0^{v_i} F(v)\, dv}{F(v_i)} = v_i - \frac{\int_0^{v_i} F(v)\, dv}{F(v_i)}.$$

That is, in equilibrium a bidder shades her bid down by an amount of $\int_0^{v_i} F(v)\, dv \;\; / F(v_i)$.

15.3 Double Auction

In many trading mechanisms, both buyers and the sellers submit bids (although the price submitted by the seller is often referred to as "ask" rather than "bid"). Such mechanisms are called double auction, where the name emphasizes that both sides of the market are competing. This section is devoted to the case when there is only one buyer and one seller. (This case is clearly about bilateral bargaining, rather than general auctions.)

Consider a Seller, who owns an object, and a Buyer. They want to trade the object through the following mechanism. Simultaneously, Seller names p_s and Buyer names p_b.

- If $p_b < p_s$, then there is no trade;
- if $p_b \geq p_s$, then they trade at price

$$p = \frac{p_b + p_s}{2}.$$

The value of the object for Seller is v_s and for Buyer is v_b. Each player knows her own valuation privately. Assume that v_s and v_b are independently and identically distributed with uniform distribution on $[0,1]$. [Recall from the first-price auction what this means.] Then, the payoffs are

$$u_b = \begin{cases} v_b - \frac{p_b+p_s}{2} & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases}$$

$$u_s = \begin{cases} \frac{p_b+p_s}{2} - v_s & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases}$$

We will now compute Bayesian Nash equilibria. In an equilibrium, one must compute a price $p_s(v_s)$ for each type v_s of the seller and a price $p_b(v_b)$ for each type v_b of the buyer. In a Bayesian Nash equilibrium, $p_b(v_b)$ solves the maximization problem

$$\max_{p_b} E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right],$$

and $p_s(v_s)$ solves the maximization problem

$$\max_{p_s} E\left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right],$$

where $E[x : A]$ is the "integral" of x on set A. (Note that $E[x : A] = E[x|A] \Pr(A)$, where $E[x|A]$ is the conditional expectation of x given A. Make sure that you know all these terms!!!)

In this game, there are many Bayesian Nash equilibria. For example, one equilibrium is given by

$$p_b = \begin{cases} X & \text{if } v_b \geq X \\ 0 & \text{otherwise} \end{cases},$$

$$p_s = \begin{cases} X & \text{if } v_s \leq X \\ 1 & \text{otherwise} \end{cases}$$

for some any fixed number $X \in [0,1]$. We will now consider the Bayesian Nash equilibrium with linear strategies.

15.3.1 Equilibrium with linear strategies

Consider an equilibrium where the strategies are affine functions of valuation, but they are not necessarily symmetric.

Step 1 *Assume that there is an equilibrium with linear strategies:*

$$
\begin{aligned}
p_b(v_b) &= a_b + c_b v_b \\
p_s(v_s) &= a_s + c_s v_s
\end{aligned}
$$

for some constants a_b, c_b, a_s, and c_s. Assume also that $c_b > 0$ and $c_s > 0$. [Notice that a and c may be different for buyer and the seller.]

Step 2 *Compute the best responses for all types.* To do this, first note that

$$p_b \geq p_s(v_s) = a_s + c_s v_s \iff v_s \leq \frac{p_b - a_s}{c_s} \tag{15.10}$$

and

$$p_s \leq p_b(v_b) = a_b + c_b v_b \iff v_b \geq \frac{p_s - a_b}{c_b}. \tag{15.11}$$

To compute the best reply for a type v_b, one first computes his expected payoff from his bid (leaving in an untegral form). As shown in Figure 15.2, the payoff of the buyer is

$$v_b - \frac{p_b + p_s(v_s)}{2}$$

when $v_s \leq v_s(p_b) = (p_b - a_s)/c_s$ and the payoff is zero otherwise. Hence, the expected payoff is

$$
\begin{aligned}
E[u_b(p_b, p_s, v_b, v_s)|v_b] &= E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right] \\
&= \int_0^{\frac{p_b - a_s}{c_s}} \left(v_b - \frac{p_b + p_s(v_s)}{2}\right) dv_s.
\end{aligned}
$$

By substituting $p_s(v_s) = a_s + c_s v_s$ in this expression, obtain

$$E[u_b(p_b, p_s, v_b, v_s)|v_b] = \int_0^{\frac{p_b - a_s}{c_s}} \left(v_b - \frac{p_b + a_s + c_s v_s}{2}\right) dv_s.$$

Visually, this is the area of the trapezoid that lies beween 0 and $v_s(p_b)$ horizontally and between the price $(p_s + p_b)/2$ and v_b vertically.[5]

[5] The area is

$$E[u_b(p_b, p_s, v_b, v_s)|v_b] = \frac{p_b - a_s}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right),$$

but it is not needed for the final result.

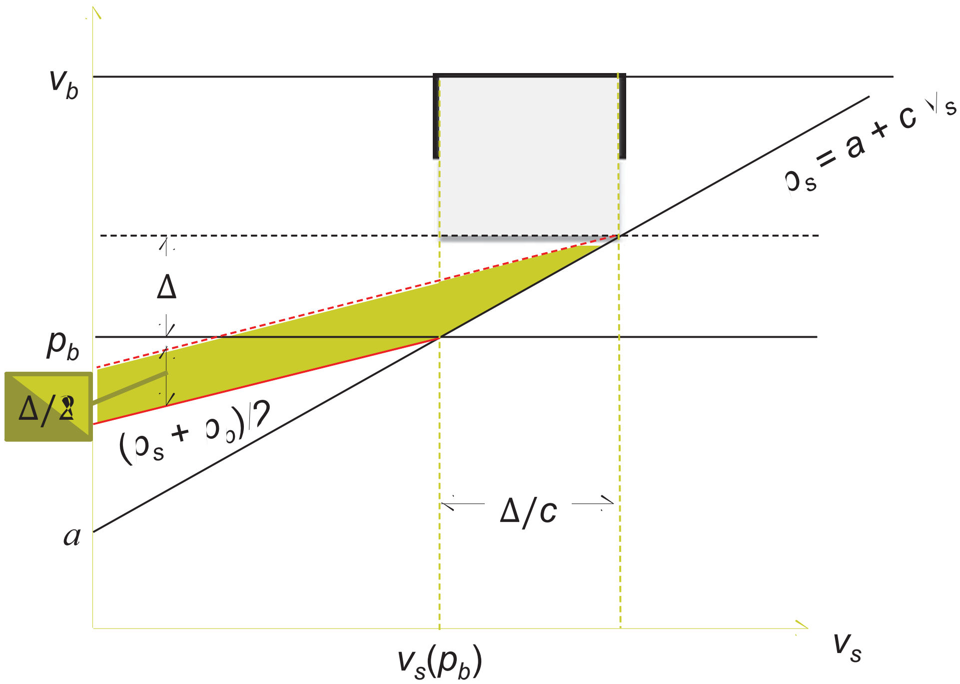

Figure 15.2: Payoff of of a buyer in double auction as his bid changes

To compute the best reply, take the derivative of the last expression with respect to p_b and set it equal to zero:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial p_b} E\left[u_b\left(p_b, p_s, v_b, v_s\right)|v_b\right] = \frac{\partial}{\partial p_b} \int_0^{\frac{p_b - a_s}{c_s}} v_b - \frac{p_b + a_s + c_s v_s}{2} \, dv_s \\
&= \frac{v_b - p_b}{c_s} - \int_0^{\frac{p_b - a_s}{c_s}} \frac{1}{2} dv_s \\
&= \frac{v_b - p_b}{c_s} - \frac{1}{2} \frac{p_b - a_s}{c_s}.
\end{aligned}
$$

Solving for p_b, obtain

$$p_b = \frac{2}{3} v_b + \frac{1}{3} a_s. \tag{15.12}$$

Graphically, a Δ amount of increase in p_b has two impacts on the expected payoff. First it causes a Δ/c_s amount of increase in $v_s(p_b)$, adding the shaded rectangular area of size $(v_b - p_b)\Delta/c_s$ in Figure 15.2. It also increases the price by an amount of $\Delta/2$, subtracting the shaded trapezoidal area of approximate size $v_s(b)\Delta/2$. At the optimum the two amounts must be equal, yielding the above equality.

Now compute the best reply of a type v_s. As in before, his expected payoff of playing p_s in equilibrium is

$$\begin{aligned} E\left[u_s\left(p_b, p_s, v_b, v_s\right)|v_s\right] &= E\left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right] \\ &= \int_{\frac{p_s - a_b}{c_b}}^{1} \left(\frac{p_s + a_b + c_b v_b}{2} - v_s\right) dv_b, \end{aligned}$$

where the last equality is by (15.11) and $p_b\left(v_b\right) = a_b + c_b v_b$. Once again, in order to compute the best reply, take the derivative of the last expression with respect to p_s and set it equal to zero:[6]

$$-\frac{1}{c_b}\left(p_s - v_s\right) + \int_{\frac{p_s - a_b}{c_b}}^{1} \frac{1}{2} dv_b = \frac{1}{2}\left(1 - \frac{p_s - a_b}{c_b}\right) - \frac{1}{c_b}\left(p_s - v_s\right) = 0.$$

Once again, a Δ increase in p_s leads to a $\Delta/2$ increase in the price, resulting in a gain of $\left(1 - \frac{p_s - a_b}{c_b}\right)\Delta/2$. It also leads to a Δ/c_b decrease in the types of buyers who trade, leading to a loss of $\left(p_s - v_s\right)\Delta/c_b$. At the optimum, the gain and the loss must be equal, yielding the above equality. Solving for p_s, one can then obtain

$$p_s = \frac{2}{3} v_s + \frac{a_b + c_b}{3}. \tag{15.13}$$

Step 3 *Verify that best replies are of the form that is assumed in Step 1.* Inspecting (15.12) and (15.13), one concludes that this is indeed the case. The important point here is to check that in (15.12) the coefficient 2/3 and the intercept $\frac{1}{3}a_s$ are constants, independent of v_b. Similarly for the coefficient and the intercept in (15.13).

Step 4 *Compute the constants.* To do this, we identify the coefficients and the intercepts in the best replies with the relevant constants in the functional form in Step 1. Firstly, by (15.12) and $p_b\left(v_b\right) = p_b$, we must have the identity

$$a_b + c_b v_b = \frac{1}{3} a_s + \frac{2}{3} v_b.$$

[6] One uses Leibnitz rule. The derivative of upper bound is zero, contributing zero to the derivative. The derivative of the lower bound is $1/c_s$, and this is multiplied by the expression in the integral at the lower bound, which is simply $p_s - v_s$. (Note that at the lower bound $p_b = p_s$, and hence the price is simply p_s.) Finally, one adds the integral of the derivative of the expression inside the integral, which is simply 1/2.

That is,

$$a_b = \frac{1}{3}a_s \tag{15.14}$$

and

$$c_b = \frac{2}{3}. \tag{15.15}$$

Similarly, by (15.13) and $p_s(v_s) = p_s$, we must have the identity

$$a_s + c_s v_s = \frac{a_b + c_b}{3} + \frac{2}{3}v_s.$$

That is,

$$a_s = \frac{a_b + c_b}{3} \tag{15.16}$$

and

$$c_s = \frac{2}{3}. \tag{15.17}$$

Solving (15.14), (15.15), (15.16), and (15.17), we obtain $a_b = 1/12$ and $a_s = 1/4$.

Therefore, the linear Bayesian Nash equilibrium is given by

$$p_b(v_b) = \frac{2}{3}v_b + \frac{1}{12} \tag{15.18}$$

$$p_s(v_s) = \frac{2}{3}v_s + \frac{1}{4}. \tag{15.19}$$

In this equilibrium, the parties trade iff

$$p_b(v_b) \geq p_s(v_s)$$

i.e.,

$$\frac{2}{3}v_b + \frac{1}{12} \geq \frac{2}{3}v_s + \frac{1}{4},$$

which can be written as

$$v_b - v_s \geq \frac{3}{2}\quad \frac{1}{4} - \frac{1}{12}\quad = \frac{3}{2}\frac{1}{6} = \frac{1}{4}.$$

Whenever $v_b > v_c$ there is a positive gain form trader. When the gain from trade is lower than 1/4, the parties leave this gain from trade on the table. This is because of the incomplete information. The parties do not know that there is a positive gain from trade. Even if they tried to find ingenious mechanisms to elicit the values, buyer would have an incentive to understate v_b and seller would have an incentive to overstate v_s, and some gains from trade would not be realized.

15.4 Investment in a Joint Project

In real life, the success of a project often requires investment by several independent parties. In an firm, production function exhibit synergies between the capital investment and the labor. A successful product development requires input from both the R&D department, who will develop the new prototype, and the marketing department, who will do market research and advertisement. In a more macro level, we need both entrepreneurs investing in new business ideas and the "workers" investing in their human capital (when they are students).

In all these examples, the return from investment for one party is increasing in the investment level by the other. For example, if R&D does not put effort in developing a good product, the market research and advertisement will be all waste. Likewise if the marketing department does not do a good job, R&D will not be useful, they will either develop the wrong product (failure in the market research) or the product will not sell because of bad advertisement. Similarly, as a student, in order to invest in your human capital (by studying rather than partying), you should anticipate that there will be jobs that will pay for your human capital, and in order for investing in skill oriented jobs, the entrepreneur should anticipate that there will be skilled people they can hire. The firms or the countries in which such investments take place proper while the others remain poor.

I will know illustrate some of the issues related to this coordination problem on a simple example. There are two players, 1 and 2, and a potential project. Each player may either invest in the project or not invest. If they both invest in the project, it will succeed; otherwise it will fail costing money to the party who invest (if there is any investment). The payoffs are as follows:Consider the payoff matrix

	Invest	Not Invest
Invest	θ, θ	$\theta - 1, 0$
Not Invest	$0, \theta - 1$	$0, 0$

Player 1 chooses between rows, and Player 2 chooses between columns. Here, the payoffs from not investing is normalized to zero. If a player invests, his payoff depends on the other player's action. If the other player also invests, the project succeeds, and both players get θ. If the other player does not invest, the project fails, and the investing player incurs a cost, totalling a net benefit of $\theta - 1$.[7] The important thing here is the return from investment is 1 utile more if the other party also invests.

Now suppose that θ is common knowledge. Consider first the case $\theta < 0$. In that case, the return from investment is so low that Invest is strictly dominated by Not Invest. (I am sure you can imagine a case in which even if you learn everything the school is offering and get the best job, it is still not worth studying.) Each player chooses not to invest. Now consider the other extreme case: $\theta > 1$. In that case, the return from investment is so high that Invest strictly dominates Not Invest, and both parties invest regardless of their expectations about the other. (For example, studying may be such a fun that you would study the material even if you thought that it will not help you get any job.) These are two extreme, uninteresting cases.

Now consider the more interesting and less extreme case of $0 < \theta < 1$. In that case, there are two equilibria in pure strategies and one equilibrium in mixed strategies. In the good equilibrium, anticipating that the other player invests, each player invests in the project, and each gets the positive payoff of θ. In the bad equilibrium, each player correctly anticipates that the other party will not invest, so that neither of them invest, yielding zero payoff for both players.

It is tempting to explain the differences between developed and underdeveloped countries that have similar resources or the successful and unsuccessful companies by such a multiple equilibria story. Indeed, it has been done so by many researchers. We will next consider the case with incomplete information and see that there are serious problems with such explanations.

Now assume that players do not know θ, but each of them gets an arbitrarily precise noisy signal about θ. In particular, each player i observes

$$x_i = \theta + \eta_i, \tag{15.20}$$

where η_i is a noise term, uniformly distributed on $[-\varepsilon, \varepsilon]$ and $\varepsilon \in (0, 1)$ is a scalar that measures the level of uncertainty players face. Assume also that θ is distributed uniformly on a large interval $[-L, L]$ where $L \gg 1 + \varepsilon$. Finally, assume that (θ, η_1, η_2) are independently distributed. We take the payoff matrix, which depends on the players'

[7] This payoff structure corresponds to investing in a project that yields 1 if the project succeeds and 0 if it fails. A player has to incur a cost c when he invests in the project. Writing $\theta = 1 - c$ for the net return from the project, we obtain the payoff structure above.

types x_1 and x_2, as

	Invest	Not Invest
Invest	x_1, x_2	$x_1 - 1, 0$
Not Invest	$0, x_2 - 1$	$0, 0$

That is, the players do not know how much the other party values the investment, but they know that the valuations are positively correlated. This is because they are both estimates about the same thing. For example, if Player 1 finds out that investment is highly valuable, i.e., x_1 is high, then he will believe that Player 2 will also find out that the investment is valuable, i.e., x_2 is high. Because of the noise terms, he will not know however what x_2 is. In particular, for $x_1 \in [0, 1]$, he will find that the other player's signal is higher than his own with probability 1/2 and lower than his own with probability 1/2:

$$\Pr(x_j < x_i | x_i) = \Pr(x_j > x_i | x_i) = 1/2. \tag{15.21}$$

This is implied by the fact that θ is uniformly distributed and we are away from the corners L and $-L$. [If you are mathematically inclined, you should prove this.]

We will now look for the symmetric Bayesian Nash equilibria in monotone (i.e. weakly increasing) strategies. A *monotone strategy* s_i here is a strategy with a cutoff value $\hat{x}_i$ such that player invests if and only if his signal exceeds the cutoff:

$$s_i(x_i) = \begin{matrix} \text{Invest} & \text{if } x_i \geq \hat{x}_i, \\ \text{Not Invest} & \text{if } x_i < \hat{x}_i. \end{matrix}$$

Any *symmetric Bayesian Nash equilibrium* s^* *in monotone strategies* has a cutoff value $\hat{x}$ such that

$$s_i^*(x_i) = \begin{matrix} \text{Invest} & \text{if } x_i \geq \hat{x}, \\ \text{Not Invest} & \text{if } x_i < \hat{x}. \end{matrix}$$

Here, symmetry means that the cutoff value $\hat{x}$ is the same for both players. In order to identify such a strategy profile all we need to do is to determine a cutoff value.

We will now find the cutoff values $\hat{x}$ that yields a Bayesian Nash equilibrium. Notice that the payoff from investment is

$$\begin{aligned} U_i(Invest, s_j | x_i) &= \Pr(s_j(x_j) = Invest | x_i)\, x_i + \Pr(s_j(x_j) = Not\ Invest | x_i)(x_i - 1) \\ &= x_i - \Pr(s_j(x_j) = Not\ Invest | x_i). \end{aligned}$$

The payoff from Not invest is simply zero. Hence, a player invests as a best reply if and only if his signal is at least as high as the probability that the other player is not investing, i.e., $x_i \geq \Pr(s_j(x_j) = Not\ Invest|x_i)$. Therefore s^* is a Bayesian Nash equilibrium iff

$$\begin{aligned}(\forall x_i \geq \hat{x}) \qquad x_i &\geq \Pr\left(s_j^*(x_j) = Not\ Invest|x_i\right) = \Pr(x_j < \hat{x}|x_i) \\ (\forall x_i < \hat{x}) \qquad x_i &\leq \Pr\left(s_j^*(x_j) = Not\ Invest|x_i\right) = \Pr(x_j < \hat{x}|x_i).\end{aligned}$$

Proof. Consider $x_i \geq \hat{x}$. According to s^*, i Invests at x_i, with expected payoff of $x_i - \Pr\left(s_j^*(x_j) = Not\ Invest|x_i\right) = x_i - \Pr(x_j < \hat{x}|x_i)$. In a Bayesian Nash equilibrium he has no incentive to deviate to Not Invest, i.e., $x_i - \Pr(x_j < \hat{x}|x_i) \geq 0$, or equivalently $x_i \geq \Pr(x_j < \hat{x}|x_i)$. Similarly, when $x_i < \hat{x}$, according to s^*, player i does Not Invest, getting 0, and hence he has no incentive to deviate to Invest and get $x_i - \Pr(x_j < \hat{x}|x_i)$ iff $x_i < \Pr(x_j < \hat{x}|x_i)$. ■

Now observe that if $x_i \geq 1$, then $x_i \geq 1 \geq \Pr(x_j < \hat{x}|x_i)$, and hence $s_i^*(x_i) = Invest$. On the other hand, if $x_i < 0$, then $x_i < 0 \leq \Pr(x_j < \hat{x}|x_i)$, and hence $s_i^*(x_i) = Not\ Invest$. Therefore, $\hat{x} \in [0, 1]$.

Most importantly, at the cutoff value the player must be indifferent between investing and not investing:

$$\hat{x} = \Pr(x_j < \hat{x}|\hat{x}). \tag{15.22}$$

Intuitively, when x_i is slightly lower than $\hat{x}$ we have $x_i \leq \Pr(x_j < \hat{x}|x_i)$, and when x_i is slightly higher than $\hat{x}$ we have $x_i \geq \Pr(x_j < \hat{x}|x_i)$. Because of continuity we must have equality at $x_i = \hat{x}$. Below, for those who want to see a rigorous proof, I make this argument more formally.

Proof. Since $\hat{x} \in [0, 1]$, there are types $x_i > \hat{x}$, and all such types invest. Hence there is a sequence of types $x_i \to \hat{x}$ with $x_i > \hat{x}$. Since each x_i invests, $x_i \geq \Pr(x_j < \hat{x}|x_i)$. Moreover, $\Pr(x_j < \hat{x}|x_i)$ is continuous in x_i. Hence, $\hat{x} = \lim x_i \geq \lim \Pr(x_j < \hat{x}|x_i) = \Pr(x_j < \hat{x}|\hat{x})$. Similarly, there are types $x_i < \hat{x}$, who do not invests, and considering such types approaching $\hat{x}$, we conclude that $\hat{x} = \lim x_i \leq \lim \Pr(x_j < \hat{x}|x_i) = \Pr(x_j < \hat{x}|\hat{x})$. Combining these two we obtain the equality. ■

Equation (15.22) shows that there is a unique symmetric Bayesian Nash equilibrium in monotone strategies.

Proposition 15.1 *There is a unique symmetric Bayesian Nash equilibrium in monotone strategies:*

$$s_i^*(x_i) = \begin{array}{ll} \textit{Invest} & \textit{if } x_i \geq 1/2, \\ \textit{Not Invest} & \textit{if } x_i < 1/2. \end{array}$$

Proof. By (15.22), we have $\hat{x} = \Pr(x_j < \hat{x}|\hat{x})$. But by (15.21), $\Pr(x_j < \hat{x}|\hat{x}) = 1/2$. Therefore,

$$\hat{x} = \Pr(x_j < \hat{x}|\hat{x}) = 1/2.$$

■

We have shown that there is a unique symmetric Bayesian Nash equilibrium in monotone strategies. It is beyond the scope of this course, but this also implies that the symmetric Bayesian Nash equilibrium is the only rationalizable strategy (with the exception of what to play at the cutoff 1/2).[8] That is the game with incomplete information has a unique solution, as opposed to the multiple equilibria in the case of complete information.

The unique solution has intuitive properties. Firstly, the investment becomes more likely when it is more valuable. This is because, as we increase θ, the probability $\Pr(x_i \geq 1/2|\theta)$ also increases. (That probability is $(\theta + \varepsilon - 1/2)/\varepsilon$ when it is in the interior $(0, 1)$.) That is, the outcome is determined by the underlying payoff parameters in an intuitive way. Secondly, the cutoff value 1/2 is also intuitive. Suppose that ε is very small so that $x_1 \cong x_2 \cong \theta$. Let us say that Invest is risk dominant if it is a best reply to the belief that the other player invests with probability 1/2 and does not invest with probability 1/2. Such beliefs are meant to be completely uninformative. Note that Invest is risk dominant if and only is $x_i \geq 1/2$. That is, the players play the risk dominant action under incomplete information.

[8] For mathematically inclined students: This is because the game is supermodular: (i) the return to investment increases with the investment of the other party and with one's own type x_i, and (ii) the beliefs are increasing in the sense that $\Pr(x_j \geq a|x_i)$ is weakly increasing in x_i. In that case, the rationalizable strategies are bounded by symmetric Bayesian Nash equilibria in monotone strategies. Clearly, when the latter is unique, there must be a unique rationalizable strategy.

15.5 Exercises with Solution

1. [Final, 2007, early exam] A consumer needs 1 unit of a good. There are n firms who can supply the good. The cost of producing the good for firm i is c_i, which is privately known by i, and $(c_1, c_2, \ldots, c_n)$ are independently and uniformly distributed on $[0,1]$. Simultaneously, each firm i sets a price p_i, and the consumer buys from the firm with the lowest price. (If $k > 1$ firms charge the lowest price, he buys from one of those firms randomly, each selling with probability $1/k$.) The payoff of i is $p_i - c_i$ if it sells and 0 otherwise.

   (a) Write this as a Bayesian game.

   Answer:

   - The set of players: $N = \{1, \ldots, n\}$, the set of firms;
   - the set of types of i: $T_i = [0,1]$, the set of possible costs c_i;
   - the set of actions of i: $A_i = [0, \infty)$, the set of possible prices p_i;
   - the utility of i:

   $$u_i(p_1, \ldots, p_n; c_1, \ldots, c_n) = \begin{cases} \frac{1}{|\{j:p_j=p_i\}|}(p_i - c_i) & \text{if } p_i \le p_j \text{ for all } j \\ 0 & \text{otherwise} \end{cases}$$

   - the beliefs: conditional on c_i, $(c_j)_{j \neq i}$ iid with uniform distribution on $[0,1]$.

   (b) Compute a symmetric, linear Bayesian Nash equilibrium. What happens as $n \to \infty$? Briefly interpret.

   Answer: See part (c)

   (c) Find all symmetric Bayesian Nash equilibrium in strictly increasing and differentiable strategies.

   [Hint: Given any $\bar{c} \in (0,1)$, the probability that $c_j \geq \bar{c}$ for all $j \neq i$ is $(1-\bar{c})^{n-1}$.]

   Answer: We are looking for an equilibrium in which each player i plays p, which is an increasing differentiable function that maps c_i to $p(c_i)$. Now, given that the other players play p, the expected utility of firm i from charging p_i at cost c_i is

$$\begin{aligned} U_i(p_i, c_i) &= \Pr(p(c_j) > p_i \text{ for all } j = i)(p_i - c_i) \\ &= \Pr \ c_j > p^{-1}(p_i) \ \text{ for all } j = i \ (p_i - c_i) \\ &= \ 1 - p^{-1}(p_i) \ ^{n-1} (p_i - c_i). \end{aligned}$$

To see the last equality, note that for all $j = i$, $\Pr(c_j > p^{-1}(p_i)) = (1 - p^{-1}(p_i))$. Since the types are *independently* distributed, we must multiply these probabilities over j—$n-1$ times. The first order condition is

$$\frac{\partial U_i}{\partial p_i} = \ 1 - p^{-1}(p_i) \ ^{n-1} - (n-1) \ 1 - p^{-1}(p_i) \ ^{n-2} (p_i - c_i) \cdot \frac{1}{p\ (c)} \ _{p(c)=p_i} = 0.$$

This equation must be satisfied at $p_i = p(c_i)$:

$$(1 - c_i)^{n-1} - (n-1)(1 - c_i)^{n-2}(p(c_i) - c_i) \frac{1}{p\ (c_i)} = 0.$$

One can rewrite this as a differential equation:

$$(1 - c_i)^{n-1} p\ (c_i) - (n-1)(1 - c_i)^{n-2} p(c_i) = -c_i (n-1)(1 - c_i)^{n-2}.$$

(If you obtain this differential equation, you will get 8 out of 10.) To solve it, notice that the left-hand side is

$$\frac{d}{dc_i} \ (1 - c_i)^{n-1} p(c_i) \ .$$

Therefore,

$$\begin{aligned} (1 - c_i)^{n-1} p(c_i) &= \quad -c_i (n-1)(1 - c_i)^{n-2} dc_i + const \\ &= \frac{n-1}{n-1}(1 - c_i)^{n-1} - \frac{n-1}{n}(1 - c_i)^n + const, \end{aligned}$$

which is obtained by changing variable to $v = 1 - c$. To have the equality at $c_i = 1$, constant must be zero. Therefore,

$$p(c_i) = 1 - \frac{n-1}{n}(1 - c_i) = \frac{1}{n} + \frac{n-1}{n} c_i.$$

This is also the symmetric linear BNE in part (b). Here, with incomplete information, the equilibrium price is a weighted average of the lowest cost and the highest possible cost. This price can be high. However, as $n \to \infty$, $p(c_i) \to c_i$, and the firm with the lowest cost sells the good at its marginal cost, as in the competitive equilibrium.

Remark 15.2 *The problem here can be viewed as a procurement auction, in which the lowest bidder wins. This is closely related to the problem in which n buyers with privately known values bid in a first-price auction.*

2. [Final 2002] Two partners simultaneously invest in a project, where the level of investment can be any non-negative real number. If partner i invests x_i and the other partner j invests x_j, then the payoff of partners i is

$$\theta_i x_i x_j - x_i^3.$$

Here, θ_i is privately known by partner i, and the other partner believes that θ_i is uniformly distributed on $[0,1]$. All these are common knowledge. Find a symmetric Bayesian Nash equilibrium in which the investment of partner i is in the form of $x_i = a + b\sqrt{\theta_i}$.

Solution: In this problem, all symmetric Bayesian Nash equilibria turn out to be of the above form; the question hints the form. I construct a Bayesian Nash equilibrium (x_1^*, x_2^*), which will be in the form of $x_i^*(\theta_i) = a + b\sqrt{\theta_i}$. The expected payoff of i from investment x_i is

$$U(x_i;\theta_i) = E\left[\theta_i x_i x_j^* - x_i^3\right] = \theta_i x_i E\left[x_j^*\right] - x_i^3.$$

Of course, $x_i^*(\theta_i)$ satisfies the first-order condition

$$0 = \partial U(x_i;\theta_i)/\partial x_i|_{x_i^*(\theta_i)} = \theta_i E\left[x_j^*\right] - 3\left(x_i^*(\theta_i)\right)^2,$$

i.e.,

$$x_i^*(\theta_i) = \sqrt{\theta_i E\left[x_j^*\right]/3} = \sqrt{\frac{E\left[x_j^*\right]}{3}}\sqrt{\theta_i}.$$

That is, $a = 0$, and the equilibrium is in the form of $x_i^*(\theta_i) = b\sqrt{\theta_i}$ where

$$b = \sqrt{\frac{E\left[x_j^*\right]}{3}}.$$

But $x_j^* = b \quad \overline{\theta_j}$, hence

$$E \quad x_j^* \quad = E\left[b \quad \overline{\theta_j}\right] = bE\left[\quad \overline{\theta_j}\right] = 2b/3.$$

Substituting this in the previous equation we obtain

$$b^2 = \frac{E \quad x_j^*}{3} = \frac{2b/3}{3} = \frac{2b}{9}.$$

There are two solutions for this equality, each yielding a distinct Bayesian Nash equilibrium. The first solution is

$$b = 2/9,$$

yielding Bayesian Nash equilibrium

$$x_i^*(\theta_i) = \frac{2}{9} \quad \overline{\theta_i}.$$

The second solution is $b = 0$, yielding the Bayesian Nash equilibrium in which each player invests 0 regradless of his type.

3. [Midterm 2, 2001] Consider the following first-price, sealed-bid auction where an indivisible good is sold. There are $n \geq 2$ buyers indexed by $i = 1, 2, \ldots, n$. Simultaneously, each buyer i submits a bid $b_i \geq 0$. The agent who submits the highest bid wins. If there are $k > 1$ players submitting the highest bid, then the winner is determined randomly among these players — each has probability $1/k$ of winning. The winner i gets the object and pays his bid b_i, obtaining payoff $v_i - b_i$, while the other buyers get 0, where $v_1, \ldots, v_n$ are independently and identically distributed with probability density function f where

$$f(x) = \begin{matrix} 3x^2 & x \in [0,1] \\ 0 & \text{otherwise.} \end{matrix}$$

   (a) Compute the symmetric, linear Bayesian Nash equilibrium.

   Answer: We look for an equilibrium of the form

$$b_i = a + cv_i$$

where $c > 0$. Then, the expected payoff from bidding b_i with type v_i is

$$\begin{aligned}
U(b_i; v_i) &= (v_i - b_i) \Pr(b_i > a + cv_j \quad \forall j = i) \\
&= (v_i - b_i) \prod_{j \neq i} \Pr(b_i > a + cv_j) \\
&= (v_i - b_i) \prod_{j \neq i} \Pr\left(v_j < \frac{b_i - a}{c} \right) \\
&= (v_i - b_i) \prod_{j \neq i} \left(\frac{b_i - a}{c} \right)^3 \\
&= (v_i - b_i) \left(\frac{b_i - a}{c} \right)^{3(n-1)}
\end{aligned}$$

for $b_i \in [a, a + c]$. The first order condition is

$$\frac{\partial U(b_i; v_i)}{\partial b_i} = -\left(\frac{b_i - a}{c} \right)^{3(n-1)} + 3(n-1) \frac{1}{c} (v_i - b_i) \left(\frac{b_i - a}{c} \right)^{3(n-1)-1} = 0;$$

i.e.,

$$-\left(\frac{b_i - a}{c} \right) + 3(n-1) \frac{1}{c} (v_i - b_i) = 0;$$

i.e.,

$$b_i = \frac{a + 3(n-1) v_i}{3(n-1) + 1}.$$

Since this is an identity, we must have

$$a = \frac{a}{3(n-1) + 1},$$

i.e., $a = 0$, and

$$c = \frac{3(n-1)}{3(n-1) + 1}.$$

(b) What happens as $n \to \infty$?

Answer: As $n \to \infty$,

$$b_i \to v_i.$$

In the limit, each bidder bids his valuation, and the seller extracts all the gains from trade.

[**Hint:** Since $v_1, v_2, \ldots, v_n$ is independently distributed, for any $w_1, w_2, \ldots, w_k$, we have

$$\Pr(v_1 \leq w_1, v_2 \leq w_2, \ldots, v_k \leq w_k) = \Pr(v_1 \leq w_1)\Pr(v_2 \leq w_2)\ldots\Pr(v_k \leq w_k).]$$

4. [Midterm 2, 2002] Consider a game between two software developers, who sell operating systems (OS) for personal computers. (There are also a PC maker and the consumers, but their strategies are already fixed.) Each software developer i, simultaneously offers "bribe" b_i to the PC maker. (The bribes are in the form of contracts.) Looking at the offered bribes b_1 and b_2, the PC maker accepts the highest bribe (and tosses a coin between them if they happen to be equal), and he rejects the other. If a firm's offer is rejected, it goes out of business, and gets 0. Let i^* denote the software developer whose bribe is accepted. Then, i^* pays the bribe b_{i^*}, and the PC maker develops its PC compatible only with the operating system of i^*. Then in the next stage, i^* becomes the monopolist in the market for operating systems. In this market the inverse demand function is given by

$$P = 1 - Q,$$

where P is the price of OS and Q is the demand for OS. The marginal cost of producing the operating system for each software developer i is c_i. The costs c_1 and c_2 are independently and identically distributed with the uniform distribution on $[0, 1]$, i.e.,

$$\Pr(c_i \leq c) = \begin{cases} 0 & \text{if } c < 0 \\ c & \text{if } c \in [0,1] \\ 1 & \text{otherwise.} \end{cases}$$

The software developer i knows its own marginal costs, but the other firm does not know. Each firm tries to maximize its own expected profit. Everything described so far is common knowledge.

(a) What quantity a software developer i would produce if it becomes monopolist? What would be its profit?

Solution: Quantity is

$$q_i = \frac{1 - c_i}{2}$$

and the profit is

$$v_i = \left(\frac{1-c_i}{2}\right)^2.$$

(b) Compute a symmetric Bayesian Nash equilibrium in which each firm's bribe is in the form of $b_i = \alpha + \gamma (1-c_i)^2$.

Solution: We have a first price auction where the valuation of buyer i, who is the software developer i, is $v_i = (1-c_i)^2/4$. His payoff from paying bribe b_i is

$$U_i(b_i; c_i) = (v_i - b_i) \Pr(b_j < b_i),$$

where

$$\begin{aligned}
\Pr(b_j < b_i) &= \Pr\left(\alpha + \gamma (1-c_j)^2 < b_i\right) = \Pr\left((1-c_j)^2 < (b_i - \alpha)/\gamma\right) \\
&= \Pr\left(1 - c_j < \sqrt{(b_i-\alpha)/\gamma}\right) = \Pr\left(c_j > 1 - \sqrt{(b_i-\alpha)/\gamma}\right) \\
&= 1 - \Pr\left(c_j \leq 1 - \sqrt{(b_i-\alpha)/\gamma}\right) = 1 - \left[1 - \sqrt{(b_i-\alpha)/\gamma}\right] \\
&= \sqrt{(b_i-\alpha)/\gamma}.
\end{aligned}$$

Hence,

$$U_i(b_i; c_i) = (v_i - b_i)\sqrt{(b_i-\alpha)/\gamma}.$$

But maximizing $U_i(b_i; c_i)$ is the same as maximizing

$$\gamma U_i(b_i; c_i)^2 = (v_i - b_i)^2 (b_i - \alpha).$$

The first order condition yields

$$2(b_i - v_i)(b_i - \alpha) + (b_i - v_i)^2 = 0,$$

i.e.,

$$2(b_i - \alpha) + (b_i - v_i) = 0,$$

i.e.,

$$b_i = \frac{1}{3} v_i + \frac{2}{3}\alpha = \frac{1}{12}(1-c_i)^2 + \frac{2}{3}\alpha.$$

Therefore,

$$\gamma = \frac{1}{12} \text{ and } \alpha = \frac{2}{3}\alpha \Longrightarrow \alpha = 0,$$

yielding

$$b_i = \frac{1}{3} v_i = \frac{1}{12}(1-c_i)^2.$$

(Check that the second derivative is $2(3b_i - 2v_i) = -2v_i < 0$.)

(c) Considering that the demand for PCs and the demand of OSs must be the same, should PC maker accept the highest bribe? (Assume that PC maker also tries to maximize its own profit. Explain your answer.)

Answer: A low-cost monopolist will charge a lower price, increasing the profit for the PC maker. Since low-cost software developers pay higher bribes, it is in the PC maker's interest to accept the higher bribe. In that case, he will get higher bribe now and higher profits later.

15.6 Exercises

1. [Midterm 2 Make Up, 2011] There are n players in a town. Simultaneously each player i contributes x_i to a public project, yielding a public good of amount

$$y = x_1 + \cdots + x_n,$$

where x_i is any real number. The payoff of each player i is

$$u_i = y^2 - c_i x_i^\gamma$$

where $\gamma > 2$ is a known parameter and the cost parameter $c_i \in \{1, 2\}$ of player i is his private information. The costs $(c_1, \ldots, c_n)$ are independently and identically distributed where the probability of $c_i = 1$ is $1/2$ for each player i.

(a) Write this formally as a Bayesian game.

(b) Find a Bayesian Nash equilibrium of this game. Verify that the strategy profile you identified is indeed a Bayesian Nash equilibrium. (If you solve this part for $n = 2$ and $\gamma = 3$, you will get 75% of the credit.)

2. [Homework 4, 2004] There are n people, who want to produce a common public good through voluntary contributions. Simultaneously, every player i contributes x_i. The amount of public good produced is

$$y = x_1 + x_2 + \cdots x_n.$$

The payoff of each player i is

$$u_i = \theta_i y - y^2 - x_i,$$

where θ_i is a parameter privately known by player i, and $\theta_1, \theta_2, \ldots, \theta_n$ are independently and identically distributed with uniform distribution on $[1, 2]$. Assume that x_i can be positive or negative. Compute a symmetric Bayesian Nash equilibrium. [Hint: symmetric means that $x_i(\theta_i) = x_j(\theta_j)$ when $\theta_i = \theta_j$. The equilibrium will be linear, in the form of $x_i(\theta_i) = a\theta_i + b$.]

3. [Homework 5, 2005] Consider a two player game with payoff matrix

	L	R
X	$3, \theta$	$0, 0$
Y	$2, 2\theta$	$2, \theta$
Z	$0, 0$	$3, -\theta$

where $\theta \in \{-1, 1\}$ is a parameter known by Player 2. Player 1 believes that $\theta = -1$ with probability 1/2 and $\theta = 1$ with probability 1/2. Everything above is common knowledge.

(a) Write this game formally as a Bayesian game.

(b) Compute the Bayesian Nash equilibrium of this game.

(c) What would be the Nash equilibria in pure strategies (i) if it were common knowledge that $\theta = -1$, or (ii) if it were common knowledge that $\theta = 1$?

4. [Homework 5, 2005] In a college there are n students. They are simultaneously sending data over the college's data network. Let $x_i \geq 0$ be the size data sent by student i. Each student i chooses x_i himself or herself. The speed of network is inversely proportional to the total size of the data, so that it takes $x_i\tau(x_1, \ldots, x_n)$ minutes to send the message where

$$\tau(x_1, \ldots, x_n) = x_1 + \cdots + x_n.$$

The payoff of student i is

$$\theta_i x_i - x_i\tau(x_1, \ldots, x_n),$$

where $\theta_i \in \{1, 2\}$ is a payoff parameter of player i, privately known by himself or herself. For each $j = i$, independent of θ_j, player j assigns probability 1/2 to $\theta_i = 1$ and probability 1/2 to $\theta_i = 2$. Everything described so far is common knowledge.

(a) Write this game formally as a Bayesian game.

(b) Compute the symmetric Bayesian Nash equilibrium of this game.

Hint: symmetric means that $x_i(\theta_i) = x_j(\theta_j)$ when $\theta_i = \theta_j$. In the symmetric equilibrium one of the types will choose zero, i.e., for some $\theta \in \{1, 2\}$, $x_i(\theta_i) = 0$ whenever $\theta_i = \theta$. The expected value $E[x_1 + \cdots + x_n]$ of $x_1 + \cdots + x_n$ is $E[x_1] + \cdots + E[x_n]$.

5. [Midterm 2, 2001] Consider the following first-price, sealed-bid auction where an indivisible good is sold. There are $n \geq 2$ buyers indexed by $i = 1, 2, \ldots, n$. Simultaneously, each buyer i submits a bid $b_i \geq 0$. The agent who submits the highest bid wins. If there are $k > 1$ players submitting the highest bid, then the winner is determined randomly among these players — each has probability $1/k$ of winning. The winner i gets the object and pays his bid b_i, obtaining payoff $v_i - b_i$, while the other buyers get 0, where $v_1, \ldots, v_n$ are independently and identically distributed with probability density function f where

$$f(x) = \begin{matrix} (\alpha+1)\, x^{\alpha} & x \in [0,1] \\ 0 & \text{otherwise} \end{matrix}$$

for some $\alpha > 0$.

(a) Compute the symmetric, linear Bayesian Nash equilibrium.

(b) What happens as $n \to \infty$, or as $\alpha \to \infty$? Give an economic explanation for each limit.

[**Hint:** Since $v_1, v_2, \ldots, v_n$ is independently distributed, for any $w_1, w_2, \ldots, w_k$, we have

$$\Pr(v_1 \leq w_1, v_2 \leq w_2, \ldots, v_k \leq w_k) = \Pr(v_1 \leq w_1) \Pr(v_2 \leq w_2) \ldots \Pr(v_k \leq w_k).]$$

6. [Midterm 2, 2001] Consider a game of public good provision in which two players simultaneously choose whether to contribute yielding payoff matrix

1\2	Contribute	Don't
Contribute	$1 - c_1$,$1 - c_2$	$1 - c_1, 1$
Don't	$1, 1 - c_2$	0,0

where the costs c_1 and c_2 are privately known by players 1 and 2, respectively. c_1 and c_2 are independently and identically distributed with uniform distribution on $[0, 2]$ (i.e., independent of his own cost, player 1 believes that c_2 is distributed uniform distribution on $[0, 2]$ and vice verse). Compute a Bayesian Nash equilibrium of this game.

7. [Final 2002 Make Up] We consider an "all-pay auction" between two bidders, who bid for an object. The value of the object for each bidder i is v_i, where v_1 and v_2 are identically and independently distributed with uniform distribution on $[0, 1]$. Each bidder simultaneously bid b_i; the bidder who bids the highest bid gets the object, and each bidder i pays his own bid b_i. (If $b_1 = b_2$, then each gets the object with probability 1/2.) The payoff of player i is

$$u_i = \begin{cases} v_i - b_i & \text{if } b_i > b_j, \\ v_i/2 - b_i & \text{if } b_i = b_j, \\ -b_i & \text{if } b_i < b_j. \end{cases}$$

Find a symmetric Bayesian Nash equilibrium in the form of $b_i = a + cv_i^2$.

8. [Homework 6, 2006] (This question is also about a game that was played in the class.) There are n students in the class. We have a certificate, whose value for each student i is v_i, where v_i is privately known by student i and $(v_1, \ldots, v_n)$ are independently and identically distributed with uniform distribution on $[0, 100]$. Simultaneously, each student i bids a real number b_i. The player who bids the highest number "wins" the certificate; if there are more than one highest bids, then we determine the "winner" randomly among the highest bidders. The winner i gets the certificate and pays b_i to the professor. [Hint: $\Pr\left(\max_{j \neq i} v_j \leq x\right) = (x/100)^{n-1}$ for any $x \in [0, 100]$.]

   (a) Find a symmetric, linear Bayesian Nash equilibrium, where $b_i(v_i) = a + cv_i$ for some constants a and c.

   (b) What is the equilibrium payoff of a student with value v_i?

   (c) Assume that $n = 80$. How much would a student with value v_i be willing to pay (in terms of lost opportunities and pain of sitting in the class) in order to play this game? What is the payoff difference between the luckiest student and the least lucky student?

9. [Homework 6, 2006] In a state, there are two counties, A and B. The state is to dump the waste in one of the two counties. For a county i, the cost of having the wasteland is c_i, where c_A and c_B are independently and uniformly distributed on $[0, 1]$. They decide where to dump the waste as follows. Simultaneously counties A and B bid b_A and b_B, respectively. The waste is dumped in the county i who bids lower, and the other county j pays b_j to i. (We toss a coin if the bids are equal. The payoff of a county is the amount of money it has minus the cost—if it contains the wasteland.)

   (a) Write this as a Bayesian game.

   (b) Find all the symmetric equilibria where the bid is a strictly increasing differentiable function of the cost. [If you can find a differential equation that characterizes the symmetric equilibria, you will get 80% of this part.]

10. [Final, 2006] Alice and Bob have inherited a factory from their parents. The value of the factory is v_A for Alice and v_B for Bob, where v_A and v_B are independently and uniformly distributed over $[0, 1]$, and each of them knows his or her own value. Simultaneously, Alice and Bob bid b_A and b_B, respectively, and the highest bidder wins the factory and pays the other sibling's bid. (If the bids are equal, we toss a coin to determine the winner.)

    (a) (5pts) Write this game as a Bayesian game.

    (b) (10 pts) Find a symmetric, linear Bayesian Nash equilibrium of this game.

    (c) (10pts) Find all symmetric Bayesian Nash equilibria of this game in strictly increasing differentiable strategies.

11. [Final 2007] There are $n \geq 2$ siblings, who have inherited a factory from their parents. The value of the factory is v_i for sibling i, where $(v_1, \ldots, v_n)$ are independently and uniformly distributed over $[0, 1]$, and each of them knows his or her own value. Simultaneously, each i bids b_i, and the highest bidder wins the factory and pays his own bid to his siblings, who share it equally among themselves. (If

the bids are equal, the winner is determined by a lottery with equal probabilities on the highest bidders.) Note that if i wins, i gets $v_i - b_i$ and any other j gets $b_i/(n-1)$.

(a) (5 points) Write this as a Bayesian game.

(b) (10 points) Compute a symmetric, linear Bayesian Nash equilibrium. What happens as $n \to \infty$? Briefly interpret.

(c) (10 points) Find all symmetric Bayesian Nash equilibrium in strictly increasing and differentiable strategies.

12. [Homework 6, 2006] There is a house on the market. There are $n \geq 2$ buyers. The value of the house for buyer i is v_i (measured in million dollars) where v_1, v_2, ..., v_n are independently and identically distributed with uniform distribution on $[0,1]$. The house is to be sold via first-price auction. This question explores whether various "incentives" can be effective in improving participation.

(a) Suppose that seller gives a discount to the winner, so that winner pays only λb_i for some $\lambda \in (0,1)$, where b_i is his own bid. Compute the symmetric Bayesian Nash equilibrium. (Throughout the question, you can assume linearity if you want.) Compute the expected revenue of the seller in that equilibrium.

(b) Suppose that seller gives a prize $\alpha > 0$ to the winner. Compute the symmetric Bayesian Nash equilibrium. Compute the expected revenue of the seller in that equilibrium.

(c) Consider three different scenarios:

- the seller does not give any incentive;
- the seller gives 20% discount ($\lambda = 0.8$);
- the seller gives \$100,000 to the winner.

For each scenarios, determine how much a buyer with value v_i is willing to pay in order to participate the auction. Briefly discuss whether such incentives can facilitate the sale of the house.

13. [Homework 6, 2006] We have a penalty kicker and a goal keeper. Simultaneously, penalty kicker decides whether to send the ball to the Left or to the Right, and

the goal keeper decides whether to cover the Left or the Right. The payoffs are as follows (where the first entry is the payoff of penalty kicker):

PK\GK	Left	Right
Left	$x-1, y+1$	$x+1, -1$
Right	$1, y-1$	$-1, 1$

Here, x and y are independently and uniformly distributed on $[-1, 1]$; the penalty kicker knows x, and the goal keeper knows y. Find a Bayesian Nash equilibrium.

14. [Final 2010] There are two identical objects and three potential buyers, named 1, 2, and 3. Each buyer only needs one object and does not care which of the identical objects he gets. The value of the object for buyer i is v_i where (v_1, v_2, v_3) are independently and uniformly distributed on $[0, 1]$. The objects are sold to two of the buyers through the following auction. Simultaneously, each buyer i submits a bid b_i, and the buyers who bid one of the two highest bids buy the object and pay their own bid. (The ties are broken by a coin toss.) That is, if $b_i > b_j$ for some j, i gets an object and pays b_i, obtaining the payoff of $v_i - b_i$; if $b_i < b_j$ for all j, the payoff of i is 0.

    (a) (5 points) Write this as a Bayesian game.

    (b) (20 points) Compute a symmetric Bayesian Nash equilibrium of this game in increasing differentiable strategies. (You will receive 15 points if you derive the correct equations without solving them.)

15. [Final 2010] A state government wants to construct a new road. There are n construction firms. In order to decrease the cost of delay in completion of the road, the government wants to divide the road into $k < n$ segments and construct the segments simultaneously using different firms. The cost of delay for the public is $C_p = K/k$ for some constant $K > 0$. The cost of constructing a segment for firm i is c_i/k where $(c_1, \ldots, c_n)$ are independently and uniformly distributed on $[0, 1]$, where c_i is privately known by firm i. The government hires the firms through the following procurement auction.

    $k+1$**st-price Procurement Auction** Simultaneously, each firm i submits a bid b_i and each of the firms with the **lowest** k bids wins one of the segments. Each winning firm is paid the lowest $k+1$st bid as the price for the construction of the segment. The ties are broken by a coin toss.

The payoff of a winning firm is the price paid minus its cost of constructing a segment, and the payoff of a losing firm is 0. For example, if $k=2$ and the bids are $(0.1, 0.2, 0.3, 0.4)$, then firms 1 and 2 win and each is paid 0.3, resulting in payoff vector $(0.3 - c_1/2, 0.3 - c_2/2, 0, 0)$.

(a) (10 points) For a given fixed k, find a Bayesian Nash equilibrium of this game in which no firm bids below its cost. Verify that it is indeed a Bayesian Nash equilibrium.

(b) (10 points) Assume that each winning firm is to pay $\beta \in (0,1)$ share of the price to the local mafia. (In the above example it pays 0.3β to the mafia and keep $0.3(1-\beta)$ for itself.) For a given fixed k, find a Bayesian Nash equilibrium of this game in which no firm bids below its cost. Verify that it is indeed a Bayesian Nash equilibrium.

(c) (5 points) Assuming that the government minimizes the sum of C_P and the total price it pays for the construction, find the condition for the optimal k for the government in parts (a) and (c). Show that the optimal k in (c) is weakly lower than the optimal k in (a). Briefly interpret the result. [Hint: the expected value of the $k+1$st lowest cost is $(k+1)/(n+1)$.]

16. [Final 2011] There are k identical objects and n potential buyers where $n > k > 1$. Each buyer only needs one object and does not care which of the identical objects he gets. The value of the object for buyer i is v_i where $(v_1, v_2, \ldots, v_n)$ are independently and uniformly distributed on $[0,1]$. The objects are sold to k of the buyers through the following auction. Simultaneously, each buyer i submits a bid b_i, and the buyers who bid one of the k highest bids buy the object and pay their own bid. (The ties are broken by a coin toss.) That is, if $b_i > b_j$ for at least $n-k$ bidders j, then i gets an object and pays b_i, obtaining the payoff of $v_i - b_i$; if $b_i < b_j$ for at least k bidders j, the payoff of i is 0.

(a) (5 points) Write this as a Bayesian game.

(b) (20 points) Compute a symmetric Bayesian Nash equilibrium of this game in increasing differentiable strategies. (You will receive 15 points if you derive the correct equations without solving them.)

Hint: Let $(x_1, \ldots, x_m)$ be independently and uniformly distributed on $[0, 1]$ and let $x_{(r)}$ be rth highest x_i among $(x_1, \ldots, x_m)$. Then, the probability density function of $x_{(r)}$ is

$$f_{m,r}(x) = \frac{m!}{r!(m-r)!}(1-x)^{r-1}x^{m-r}.$$

17. [Final 2011] Consider the following charity auction. There are two bidders, namely 1 and 2. Each bidder i has a distinct favored charity. Simulatenously, each bidder i contributes b_i to the auction. The highest bidder wins, and the sum $b_1 + b_2$ goes to the favored charity of the winner. The winner is determined by a coin toss in case of a tie. The payoff of the bidder i is

$$u_i(b_1, b_2, \theta_i) = \begin{array}{rl} \theta_i(b_1 + b_2) - b_i^\gamma & \text{if } i \text{ wins} \\ -b_i^\gamma & \text{otherwise,} \end{array}$$

where $\gamma > 1$ is a known parameter, θ_i is privately known by player i, and θ_1 and θ_2 are independently and uniformly distributed on $[0, 1]$. Find a differential equation that must be satisfied by strategies in a symmetric Bayesian Nash equilibrium. (Assume that the equilibrium strategies are increasing and differentiable.)

18. [Homework 5, 2011] Consider an n-player game in which each player i selects a search level $s_i \in [0, 1]$ (simultaneously), receiving the payoff

$$u_i(s_1, \ldots, s_n, \theta_1, \ldots, \theta_n) = \theta_i s_1 \cdots s_n - s_i^\gamma/\gamma,$$

where $(\theta_1, \ldots, \theta_n)$ are independently and identically distributed on $[0, \infty)$. the expected value of each Here, $\gamma > 1$ is commonly known and θ_i is privately known by player i. (Denote the expected value of θ_i by $\bar{\theta}$ and the expected value of θ_i^α by $\bar{\theta}_\alpha$ for any $\alpha > 0$.)

(a) For $\gamma = 2$, find the symmetric linear Bayesian Nash equilibria.

(b) For $n = \gamma = 2$, find the symmetric Bayesian Nash equilibria.

19. [Homework 5, 2011] Consider an n-player first price auction in which the value of the object auctioned is v_i for player i, where $(v_1, \ldots, v_n)$ are independently and identically distributed with CDF F where $F(v) = v^\alpha$ for some $\alpha > 0$. The value of v_i is privately known by player i. Compute a symmetric Bayesian Nash equilibrium.

20. [Homework 5, 2011] Consider an auction with two buyers where the value of the object auctioned is v_i for player i, where (v_1, v_2) are independently and identically distributed with uniform distribution on $[0, 1]$. The value of v_i is privately known by player i. In the auction, the buyers simultaneously bid b_1 and b_2 and the highest bidder wins the object and pays the average bid $(b_1 + b_2)/2$ as the price. The ties are broken with a coin toss. Compute a symmetric Bayesian Nash equilibrium.