Homework #3 Solutions

Problem 1

There are two subgames, or stages. At stage 1, each ice cream parlor i (I call it firm i from now on) selects location x_i simultaneously. At stage 2, each firm i chooses prices p_i. To find SPE, we start from stage 2.

At stage 2, $(x_1,\ x_2)$ are given. If $x_1 = x_2$, whoever charges less gets the whole consumers and if their prices are same, each get half of the consumers. Thus, only possible Nash equilibrium is $p_1 = p_2 = 0$, as setting any positive price would make the other firm to undercut slighly (your price - ϵ) and take all the kids; and you want to undercut the other firm slightly to take all the kids back.

Consider the case $x_1 \neq x_2$. Without the loss of generality, I assume $x_1 < x_2$. Note that NE is only possible when we have an interior solution: there is a "mid-point" t such that $x_1 < t < x_2$ and kid at t is indifferent. If we do not have interior solution, it means that kids at $[0,\ x_1]$ or $[x_2,\ 1]$are indifferent and others prefer one firm, or all kids go to one firm, say firm i. For the latter case, firm j would deviate to a slightly lower price so that it can make positive profit. For the former case, firm j would deviate to a slightly lower price so that it can sell to $[0,\ x_1]$ and some more, instead of selling only to $\frac{x_1}{2}$ kids (this case is for $j = 1$; $j = 2$ case is similar).

For the interior solution case, we have

$$c\left(x_1 - t\right)^2 + p_1 = c\left(x_2 - t\right)^2 + p_2$$

Solving, we get

$$t = \frac{p_2 - p_1}{2c\left(x_2 - x_1\right)} + \frac{x_1 + x_2}{2}$$

Firm 1 solves the following profit maximization problem:

$$Max_{p_1} t p_1 = \left\{\frac{p_2 - p_1}{2c\left(x_2 - x_1\right)} + \frac{x_1 + x_2}{2}\right\} p_1$$

Taking FOC (first order condition), we have

$$p_1^{BR}(p_2) = \frac{p_2}{2} + \frac{c\left(x_2^2 - x_1^2\right)}{2}$$

Similarly, firm 2 solves

$$Max_{p_2} (1 - t) p_2 = \left\{1 - \frac{p_2 - p_1}{2c\left(x_2 - x_1\right)} + \frac{x_1 + x_2}{2}\right\} p_2$$

Taking FOC, we have

$$p_2^{BR}(p_1) = c\left(x_2 - x_1\right) + \frac{p_1}{2} - \frac{c\left(x_2^2 - x_1^2\right)}{2}$$

Solving two equations, we get NE:

$$p_1 = \frac{c}{3}\left(x_2 - x_1\right)\left(x_1 + x_2 + 2\right),\ p_2 = \frac{c}{3}\left(x_2 - x_1\right)\left(4 - x_1 - x_2\right)$$

Profits are

$$\pi_1 = \frac{c}{18}(x_2 - x_1)(x_1 + x_2 + 2)^2, \ \pi_2 = \frac{c}{18}(x_2 - x_1)(4 - x_1 - x_2)^2$$

At stage 1, choosing $x_1 = x_2$ cannot be an equilibrium, as in this case, they will make zero profit at the stage 2 since $p_1 = p_2 = 0$ is NE in the stage 2. Firm 1 knows that his profit will be $\pi_1 = \frac{c}{18}(x_2 - x_1)(x_1 + x_2 + 2)^2$ given x_2. Taking FOC, we have

$$\frac{\partial \pi_1}{\partial x_1} = \frac{c}{18}(x_1 + x_2 + 2)(x_2 - 3x_1 - 2) < 0$$

as $0 \leq x_1 < x_2 \leq 1$. Thus, choosing $x_1 = 0$ is optimal. Similiarly, for firm 2,

$$\frac{\partial \pi_2}{\partial x_2} = \frac{c}{18}(4 + x_1 - 3x_2)(4 - x_1 - x_2) > 0$$

Therefore, choosing $x_2 = 1$ is optimal.

In summary, SPE is that at stage 1, firm 1 chooses $x_1 = 0$ and firm 2 chooses $x_2 = 1$. For stage 2, if $x_1 = x_2$, $p_1 = p_2 = 0$. If $x_1 \neq x_2$, $p_1 = \frac{c}{3}(x_2 - x_1)(x_1 + x_2 + 2)$, $p_2 = \frac{c}{3}(x_2 - x_1)(4 - x_1 - x_2)$.

Problem 2

I denote choices as P1 chooses L or R, P2 chooses l or r, P1 chooses A or B, P2 chooses a or b (from top to bottom order). The bottom subgame has three NE: $(A, a), (B, b), \left(\frac{1}{4}A + \frac{3}{4}B, \ \frac{1}{4}a + \frac{1}{4}b\right)$.

At the next subgame, If the first NE is played, payoff is 3 for both players so P2 will choose r. Otherwise, payoff is less than 2 (1 for (B, b) and $\frac{3}{4}$ for $\left(\frac{1}{4}A + \frac{3}{4}B, \ \frac{1}{4}a + \frac{1}{4}b\right)$), so P2 will choose l.

For the next (and last) subgame, if (A, a) is played and P2 played r, P1 will choose R. Otherwise, he is indifferent between L and R.

Therefore, SPE are $(RA, \ ra)$, $(\{pL + (1-p)R\} B, \ lb)$, $\left(\{pL + (1-p)R\} \left\{\frac{1}{4}A + \frac{3}{4}B\right\}, \ \left\{l, \frac{1}{4}a + \frac{1}{4}b\right\}\right)$.

Problem 3

The proposed strategy is for player i to offer δp_i to other players and $1 - \delta(1 - p_i)$ for himself, and to accept an offer if it is at least p_i. To check that this strategy is a SPE, we consider all single deviations.

Suppose i is the proposer after any history. His equilibrium strategy has payoff $1 - \delta(1 - p_i)$. He can deviate to any strategy which offers all other players $k_j \geq \delta p_j$, which would be accepted, yielding a payoff of $1 - \delta \sum k_j \leq 1 - \delta(1 - p_i)$. This is not a profitable deviation. He can deviate to offer some player less, $k_j < p_j$ for some j. Then the offer is rejected, and in the next round he has expected payoff $\delta p_i < 1 - \delta(1 - p_i)$. This is not a profitable deviation.

Next, consider i as a non-proposer after any history. Suppose he is offered $k_i \geq \delta p_i$. If he accepts the offer, he gets either k_i if all other non proposers accept, or expected δp_i in the next round. He can deviate to reject the offer, and he would get an expected payoff δp_i in the next round. Thus, deviating is not profitable.

If a non-proposer is offered $k_i < \delta p_i$, he should reject in equilibrium, for an expected payoff of δp_i in the next round. If he deviates to accept, he gets either k_i if all other players accept, or δp_i if someone reject. Both cases are weakly worse than rejecting, so deviating is not profitable.

We have checked all possible single deviations and none are profitable.

Problem 4

For this problem, let us define $\pi^t(1)$ to be the proposer in period t and $\pi^t(2), \pi^t(3)$ to be the first and second responders in period t. Let e_i be the proposal that gives 1 to player i and 0 to everyone else. Define x^t to be the proposal at period t and x_i^t be i's share of that.

The states are: $\{k_0, p(1), p(2), p(3)\}$.

For any division $a = (p_1, p_2, 1-p_1-p_2)$ where each player has strictly positive payoff, there is some δ where a SPE gives division a. Consider the following strategy:

At $t = 0$, we start in state k_0. In this state, player $\pi^0(1)$ proposes $x^0 = a$. $\pi^0(2)$ accepts iff $x^0 = a$. $\pi^0(3)$ accepts if $x^0 = a$ or $x^0_{\pi^0(3)} \geq \delta$.

After k_0, if $x^0 = a$ we go to stage $p(\pi^0(1))$. Otherwise, if $\pi^0(2)$ rejected, we go to stage $p(\pi^0(2))$. If $\pi^0(2)$ accepts and $\pi^0(3)$ rejected, we go to stage $p(\pi^0(3))$.

In stage $p(i)$, the proposer offers $x^t = e_i$. In these states, $\pi^t(2)$ accepts if the proposer offers $x^t = e_i$. $\pi^t(3)$ accepts if $x^t_{\pi^t(3)} \geq \delta$ OR $x^t = e_i$.

After rejection in state $p(i)$, we stay in $p(i)$ if $x^t = e_i$. Otherwise, if $\pi^t(2)$ rejects, we go to $p(\pi^t(2))$. If $\pi^t(2)$ accepts and $\pi^t(3)$ rejects, we go to $p(\pi^t(2))$.

To check that this is an equilibrium, we look at all single deviations after any valid history. We first check that there are no profitable deviations at $t = 0$.

Proposer: At $t = 0$, $\pi^0(1)$ offers $x^t = a$. If he deviates and offers $x^0 \neq a$, it is rejected by $\pi^0(2)$. Then in the next period we are in stage $p(\pi^0(2))$, where $e_{\pi^0(2)}$ is offered and accepted. This gives payoff 0 to player $\pi^0(1)$, which is not a profitable deviation.

Next, we look at deviations by the responder at $t = 0$.

Case: $x^0 = a$. In equilibrium both responders accept. If either responder chooses to reject, we go to state $p_(\pi^0(1))$, where both responders get payoff 0. This is clearly not a profitable deviation.

Case: $x^0 \neq a$ and $x^0_{\pi^0(3)} \geq \delta$. Consider $\pi^0(3)$. Assume $\pi^0(2)$ accepts. In equilibrium, $\pi^0(3)$ accepts and gets $x^0_{\pi^0(3)} \geq \delta$. If he deviates and rejects, we go to state $p(\pi^0(3))$, which leads to payoff δ, so deviation is not profitable. After the history where $\pi^0(2)$ does not accept, $\pi^0(3)$ is indifferent between his actions.

Consider $\pi^0(2)$. In equilibrium, $\pi^0(2)$ rejects the offer, we go to state $p(\pi^0(2))$ which yields payoff δ. If player $\pi^0(2)$ instead deviates and accepts, we know that $\pi^0(3)$ also accepts, which yields payoff of $x^0_{\pi^0(2)}$ for $\pi^0(2)$. Since we assume that δ is large and $x^t_{\pi^0(3)} \geq \delta$, $x^0_{\pi^0(2)}$ must be small, and thus less than δ. This requires $\delta \geq 0.5$.

Case: $x^0 \neq a$ and $x^0_{\pi^0(3)} < \delta$. Consider $\pi^0(3)$. Assume $\pi^0(2)$ accepts. Player $\pi^0(3)$ rejects in equilibrium and gets δ in state $p(\pi^0(3))$. If $\pi^0(3)$ accepts, he instead gets $x^0_{\pi^0(3)} < \delta$, which is not profitable. if $\pi^0(2)$ rejects, $\pi^0(3)$ is indifferent again.

Consider $\pi^0(2)$. In equilibrium, $\pi^0(2)$ rejects and gets δ. If he deviates and accepts, $\pi^0(3)$ rejects and we go to $p(\pi^0(3))$, where $\pi^0(2)$ gets 0.

Next, we consider $t \geq 1$, when we are in state $p(i)$.

To see that the proposer has no profitable deviation, consider two cases.

If we are not in state $p(\pi^t(1))$: In equilibrium the proposer offers e_i and is accepted, so the proposer has payoff 0. If he offers any other $x^t \neq e_i$, the offer is rejected and we go to $p(\pi^t(2))$, where the proposer has payoff 0. The proposer is indifferent between any offer.

If we are in the state $p(\pi^t(1))$: In equilibrium, $\pi^t(1)$ offers $x^t = e_{\pi^t(1)}$, it is accepted and he has payoff 1. This is his maximum payoff, so there is no profitable deviation.

Finally, we have to check that responders have no profitable deviation, in three cases.

1. $x^t = e_i$
2. $x^t \neq e_i$ and $x^t_{\pi^t(3)} \geq \delta$
3. $x^t \neq e_i$ and $x^t_{\pi^t(3)} < \delta$

Case 1: In equilibrium all responders accept x^t. By deviating to rejection (for either player), we return to this state again in the next period, and all players are weakly worse off.

Case 2: First consider $\pi^t(3)$. Suppose that $\pi^t(2)$ accepts. In equilibrium, $\pi^t(3)$ accepts and gets a payoff of $\delta^t x^t_{\pi^t(3)}$. If he deviates and rejects, we go to $p(\pi^t(3))$, where he gets a payoff δ^{t+1}, which is not a profitable deviation. If $\pi^t(2)$ rejected, $\pi^t(3)$ is indifferent.

Now consider $\pi^t(2)$. In equilibrium, he rejects, and we go to $p(\pi^t(2))$, where he gets δ^{t+1}. If he deviates and accepts, he instead gets $\delta^t x^t_{\pi^t(2)}$. Because δ is large and $x^t_{\pi(3)} \geq \delta$, we can assume that $\delta^t x^t_{\pi^t(2)} \leq \delta^{t+1}$.

Case 3: First consider $\pi^t(3)$. Suppose that $\pi^t(2)$ accepts. In equilibrium, $\pi^t(3)$ rejects and gets payoff δ^{t+1}. If he deviates and accepts, he gets $\delta^t x^t_{\pi^t(3)}$ strictly less. This is not a profitable deviation. If $\pi^t(2)$ rejected, $\pi^t(3)$ is indifferent.

Now consider $\pi^t(2)$. In equilibrium he rejects, and we go to $p(\pi^t(2))$, where the payoff is δ^{t+1}. If instead he deviates and accepts, $\pi^t(3)$ rejects, we go to $p(\pi^t(3))$ and $\pi^t(2)$ gets payoff 0. This is not profitable.

Thus, we see that there is no single profitable deviation, and we have a subgame perfect equilibrium.

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