14.123 Problem Set 2 Solution
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Q1. There are two urns, A and B, each consisting of 100 balls, some are black and some are red. In urn A there are 30 red balls, but the number of red balls in urn B is not known. We draw a ball from urn A with color α and a ball from urn B with color β. Consider the following acts:

$$f_{A,r} = \begin{cases} 100 & \text{if } \alpha\text{=red} \\ 0 & \text{if } \alpha\text{=black} \end{cases} \qquad f_{A,b} = \begin{cases} 0 & \text{if } \alpha\text{=red} \\ 100 & \text{if } \alpha\text{=black} \end{cases}$$
$$f_{B,r} = \begin{cases} 110 & \text{if } \beta\text{=red} \\ 0 & \text{if } \beta\text{=black} \end{cases} \qquad f_{B,b} = \begin{cases} 0 & \text{if } \beta\text{=red} \\ 110 & \text{if } \beta\text{=black} \end{cases}$$

Let c be the choice function induced by $\succeq$. Find the sets $c(\{f_{A,r}, f_{A,b}, f_{B,r}, f_{B,b}\})$ that are consistent with $110 \succ 100 \succ 0$ and Savages postulates.

In the terminology of Lecture 3, the set of states is

$$\{(\alpha, \beta)\} = \{(r, r), (r, b), (b, r), (b, b)\},$$

and the set of consequences is $C = \{0, 100, 110\}$. Under Savage's postulates, there exists a utility function $u : C \to \mathbb{R}$ and a probability measure $p : 2^S \to [0, 1]$ such that

$$f \succeq g \iff \sum_{c \in C} p(\{s|f(s) = c\})u(c) \geq \sum_{c \in C} p(\{s|g(s) = c\})u(c).$$

From $110 \succ 100 \succ 0$, we have $u(110) > u(100) > u(0)$. For any probability measure p, we have

$$\begin{aligned} &0.7 * u(100) + 0.3 * u(0) \\ =&\sum_{c \in C} p(\{s|f_{A,b}(s) = c\})u(c) \\ >&\sum_{c \in C} p(\{s|f_{A,r}(s) = c\})u(c) \\ =&0.3 * u(100) + 0.7 * u(0), \end{aligned}$$

and $f_{A,b} \succ f_{A,r}$.

On the other hand, the expected utility from $f_{B,r}$ and $f_{B,b}$ are

$$\sum_{c \in C} p(\{s|f_{B,r}(s) = c\})u(c) = p(\beta = \text{red})u(110) + p(\beta = \text{black})u(0),$$
$$\sum_{c \in C} p(\{s|f_{B,b}(s) = c\})u(c) = p(\beta = \text{red})u(0) + p(\beta = \text{black})u(110),$$

respectively.

The preference among $f_{A,b}, f_{B,r}, f_{B,b}$ depends on the probability measure and the utility function, and the possible choice sets $c(\{f_{A,r}, f_{A,b}, f_{B,r}, f_{B,b}\})$ are

$$\{f_{A,b}\}, \{f_{B,r}\}, \{f_{B,b}\}, \{f_{A,b}, f_{B,r}\}, \{f_{A,b}, f_{B,b}\}, \{f_{B,r}, f_{B,b}\}, \{f_{A,b}, f_{B,r}, f_{B,b}\}.$$

The following is the example of p and u for each choice set:

$$\begin{aligned}
\{f_{A,b}\} &: u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.5 \\
\{f_{B,r}\} &: u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.7 \\
\{f_{B,b}\} &: u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.3 \\
\{f_{A,b}, f_{B,r}\} &: u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.63 \\
\{f_{A,b}, f_{B,b}\} &: u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.37 \\
\{f_{B,r}, f_{B,b}\} &: u(0) = 0, u(100) = 0, 7, u(110) = 1, p = 0.5 \\
\{f_{A,b}, f_{B,r}, f_{B,b}\} &: u(0) = 0, u(100) = 1, u(110) = 1.4, p = 0.5
\end{aligned}$$

Q2. (6.C.19 in MWG) Suppose that an individual has a Bernoulli utility function $u(x) = -e^{-\alpha x}$ where $\alpha > 0$. His (nonstochastic) initial wealth is given by w. There is one riskless asset and there are N risky assets. The return per unit invested on the riskless asset is r. The returns of the risky assets are independent and normally distributed with means $\mu = (\mu_1, \cdots, \mu_N)$. Derive the demand function for these $N + 1$ assets.

Let $(\sigma_1^2, \cdots, \sigma_N^2)$ be the variances of the risky assets. When the portfolio is $(\alpha_0, \cdots, \alpha_N)$ with $\sum \alpha_i = 1$, the expected return is

$$\begin{aligned}
&\mathbb{E}[-\exp(-\alpha w(\alpha_0, \cdots, \alpha_N)'(r, \cdots, r_N))] \\
=& -\exp(-\alpha w(\alpha_0 r + \sum_{i>0} \alpha_i(\mu_i - \frac{1}{2}\alpha w \alpha_i \sigma_i^2))).
\end{aligned}$$

The expected return is maximized when

$$\alpha_0 r + \sum_{i>0} \alpha_i(\mu_i - \frac{1}{2}\alpha w \alpha_i \sigma_i^2)$$

is maximized, and the constraint is

$$\sum \alpha_i = 1.$$

We have

$$\frac{\partial}{\partial \alpha_i} : -r + \mu_i - \alpha w \alpha_i \sigma_i^2 = 0,$$

and

$$\alpha_i = \frac{\mu_i - r}{\alpha w \sigma_i^2}.$$

Q3. (6.D.3 in MWG) Verify that if a distribution $G(\cdot)$ is an elementary increase in risk from a distribution $F(\cdot)$, then $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.

Let $G(\cdot)$ be an elementary increase from $F(\cdot)$ on the interval $[x', x'']$, and define $I(x) = \int_{x'}^{x} [F(t) - G(t)]dt$. $I(x') = 0$, and by the definition of G, $I(x'') = 0$, $I(x) \leq 0, \forall x \in [x', x'']$.

$$\int_{x'}^{x''} u(x) d(F(x) - G(x)) = -\int_{x'}^{x''} u'(x)(F(x) - G(x))dx = \int_{x'}^{x''} u''(x) I(x) dx,$$

and together with $u'' < 0$,

$$\int_{x'}^{x''} u(x) d(F(x) - G(x)) \geq 0$$

for any nondecreasing concave function u.

Specifically, define $G(\cdot)$ as

$$G(x) = \begin{cases} F(x) & \text{if } x \notin [x', x'') \\ \frac{\int_{x'}^{x''} F(x)dx}{x'' - x'} & \text{if } x \in [x', x''). \end{cases}$$

This corresponds to $y \sim G, x \sim F, y = x + z$ with

$$z|x = \begin{cases} x' - x & \text{with probability } \frac{x'' - x}{x'' - x'} \\ x'' - x & \text{with probability } \frac{x - x'}{x'' - x'}. \end{cases}$$

Q4. Consider a monopolist who faces a stochastic demand. If he produces q units, he incurs a zero marginal cost and sells the good at price $P(\theta, q)$ where $\theta \in [\underline{\theta}, \bar{\theta}]$ is an unknown demand shock where P and C twice differentiable. Assume that the profit function is strictly concave in q for each given θ, and $P(\theta, q) + qP_q(\theta, q)$ is increasing in θ, where P_q is the derivative of P with respect to q. The monopolist is expected profit maximizer.

(a) Show that there exists a unique optimal production level q^*.

(b) Show that if the distribution of θ changes from G to F where F first-order stochastically dominates G, then the optimal production level q^* weakly increases.

(c) Take $P(\theta, q) = \phi(\theta) - \gamma(q)$. Suppose that there are two identical monopolists as above in two independent but identical markets. Find conditions under which the monopolists have a strict incentive to merge and share the profit from each market equally.

(a) Given the zero marginal cost, the monopolist maximizes $\int qP(\theta, q)dF(\theta)$. The profit function is strictly concave in q for every θ

$$\iff \frac{\partial^2}{\partial q^2}(qP(\theta, q)) < 0 \ \forall \theta, q,$$

and we have

$$\int \frac{\partial^2}{\partial q^2}(qP(\theta, q))dF(\theta) < 0.$$

The maximization problem is strictly concave, and there exists a unique optimum q^*.

(b) Let q_G and q_F be the optimum for G and F, respectively. We have

$$\int (P(\theta, q_G) + q_G P_q(\theta, q_G))dG(\theta) = 0.$$

Since $P(\theta, q) + qP_q(\theta, q)$ is increasing in θ, when F first-order stochastically dominates G,

$$\begin{aligned} 0 &= \int (P(\theta, q_F) + q_F P_q(\theta, q_F))dF(\theta) \\ &= \int (P(\theta, q_G) + q_G P_q(\theta, q_G))dG(\theta) \\ &\leq \int (P(\theta, q_G) + q_G P_q(\theta, q_G))dF(\theta). \end{aligned}$$

By the concavity of the maximization problem, $\int (P(\theta, q) + qP_q(\theta, q))dF(\theta)$ is strictly decreasing in q, and the optimum for F weakly increases.

(c) If two monopolists share the profit equally, their expected profit is

$$\begin{aligned} &\frac{1}{2} \max_{q_1, q_2} \mathbb{E}[q_1(\phi(\theta_1) - \gamma(q_1)) + q_2(\phi(\theta_2) - \gamma(q_2))] \\ =&\frac{1}{2} \max_{q_1, q_2} ((q_1 + q_2)\mathbb{E}[\phi(\theta)] - q_1\gamma(q_1) - q_2\gamma(q_2)). \end{aligned}$$

The profit function is concave in q, which implies that $-q\gamma(q)$ is concave in q. By Jensen's inequality, the optimal q_1 is the same as q_2. Let $q = q_1 + q_2$, then

$$\begin{aligned} &\frac{1}{2}\max_{q_1,q_2}\mathbb{E}[q_1(\phi(\theta_1)-\gamma(q_1))+q_2(\phi(\theta_2)-\gamma(q_2))] \\ =&\frac{1}{2}\max_{q}(q\mathbb{E}[\phi(\theta)]-q\gamma(\frac{q}{2})) \\ =&\max_{q}(\frac{q}{2}\mathbb{E}[\phi(\theta)]-\frac{q}{2}\gamma(\frac{q}{2})), \end{aligned}$$

and the monopolists choose the same quantity as before. They will never have a strict incentive to merge.

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