Problem Set 1 - Solutions

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Question (i)

Part (a)

Set $\bar{X} = \sum_{i=1}^n X_i$, aggregate endowment. From the lecture notes, an optimal risk sharing contract $(Y_1, \ldots, Y_n)$ is given by

$$Y_i = \frac{1/\alpha_i}{1/\alpha_1 + \ldots + 1/\alpha_n}\bar{X} + \tau_i \quad \forall i = 1, \ldots, n,$$

where $\tau_1, \ldots, \tau_n \in \mathbb{R}$ are state-independent transfers such that $\sum_{i=1}^n \tau_i = 0$. Agent i gets

$$E[u_i(Y_i)] = -E\left[e^{-\frac{1}{1/\alpha_1+\ldots+1/\alpha_n}\bar{X}}\right]e^{-\alpha_i\tau_i} = -e^{-\alpha_i\left(\frac{1/\alpha_i}{1/\alpha_1+\ldots+1/\alpha_n}E[\bar{X}]-\frac{1/\alpha_i}{2(1/\alpha_1+\ldots+1/\alpha_n)^2}Var(\bar{X})+\tau_i\right)}.$$

Hence

$$Y_i \sim \frac{1/\alpha_i}{1/\alpha_1 + \ldots + 1/\alpha_n}E[\bar{X}] - \frac{1/\alpha_i}{2(1/\alpha_1 + \ldots + 1/\alpha_n)^2}Var(\bar{X}) + \tau_i,$$

where the right-hand side is the certainty equivalent of Y_i for agent i.

Part (b)

Notice that $E[\bar{X}] = n\mu$ and

$$Var(\bar{X}) = \sum_{i=1}^n\sum_{j=1}^n Cov(X_i, X_j) = \sum_{i=1}^n Var(X_i) + \sum_{i=1}^n\sum_{j\neq i} Cov(X_i, X_j) = n\sigma^2(1 + (n-1)\rho).$$

The results in Part (a) become

$$\begin{aligned}
Y_i &= \frac{1}{n}\bar{X} + \tau_i,\\
E[u_i(Y_i)] &= -e^{-\alpha(\frac{1}{n}E[\bar{X}]-\frac{\alpha}{2n^2}Var(\bar{X})+\tau_i)} = -e^{-\alpha(\mu-\frac{\alpha}{2n}\sigma^2(1+(n-1)\rho)+\tau_i)},\\
Y_i &\sim \mu - \frac{\alpha}{2n}\sigma^2(1 + (n-1)\rho) + \tau_i.
\end{aligned}$$

Summing up the agents' certainty equivalent, the society as a whole is willing to pay

$$n\mu - \frac{\alpha}{2}\sigma^2(1 + (n-1)\rho)$$

for the assets. If we focus on symmetric contracts, $\tau_1 = \ldots = \tau_n = 0$. Therefore

$$(\sigma^2, \rho) \gtrsim (\tilde{\sigma}^2, \tilde{\rho}) \quad \Leftrightarrow \quad \sigma^2(1 + (n-1)\rho) \leqslant \tilde{\sigma}^2(1 + (n-1)\tilde{\rho}),$$

which is the agent's preference over the assets. Comments: (i) fixing ρ, the agent prefers lower σ^2, and (ii) fixing σ^2, the agents prefers lower ρ.[1] Intuition: $Var(\bar{X})$ (and therefore $Var(Y_i)$) is decreasing in σ^2 and ρ, and the agent is risk averse.

Question (ii)

See solution to question 2, pset 1 2010.

Question (iii)

See solution to question 2, pset 1 2014.

Question (iv)

See solution to question 2, final 2011.

[1] Notice that $(1 + (n-1)\rho) \geqslant 0$ by assumption, otherwise the assets are not *jointly* normal, that is, Σ is not positive semi-definite. In fact, $1 + (n-1)\rho$ is an eigenvalue of Σ.