Recitation 2: Exponential vs. Quasi-Hyperbolic Discounting

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Exponential Discounting Model

$$U_t \equiv \sum_{\tau=t}^{\infty} \delta^{\tau-t} u_\tau = u_t + \delta u_{t+1} + \delta^2 u_{t+2} + \delta^3 u_{t+3} + \dots$$

- What is the key assumption of this model?
  - Amount of patience between any two periods the same
- What does this assumption imply?
  - Same degree of patience in the short- and long-run
  - Time consistency
  - No demand for commitment
- Does this seem realistic?

Exponential discounting: calibration

- Assume exponential discounting and linear utility of consumption.

- A student is indifferent between \$100 today and \$120 in two weeks.
  - What is δ? $5/6$ for two weeks.

$$100 = \frac{5}{6} \cdot 120$$

  - So the student discounts one month by $(5/6)^2$.
  - Discounts one year by $(5/6)^{24}$.

- Implies indifference between \$100 today and \$7949.68 in one year!

$$100 = \left(\frac{5}{6}\right)^{24} \cdot 7949.68$$

Exponential discounting: calibration

- Assume exponential discounting and linear utility of consumption.
- Suppose $\delta = 0.9$ (over one month).
- Pick between \$50 today and \$100 in two months.
  - Will pick \$100 in two months. $100 \cdot 0.9^2 = 81 > 50$.
- Suppose $\delta = 0.7$.
- Pick between \$50 today and \$100 in two months.
  - Will pick \$50 today. $100 \cdot 0.7^2 = 49 < 50$.

Evidence against the Exponential Discounting Model

- Short-run impatience and long-run patience
- Time inconsistency
- Demand for commitment

Evidence against the Exponential Discounting Model

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What Does Patience being Constant over Time Mean?

- Question 1: would you like to
  (a) eat one piece of candy now, or
  (b) eat two pieces of candy in an hour?

- Question 2: would you like to
  (a) eat one piece of candy in a week, or
  (b) eat two pieces of candy in a week and an hour?

- Patience being constant over time means you'd either choose (a) for both or (b) for both

- Bonus question: why do the (a) options have one piece and the (b) options have two pieces?
  - The exponential discounting world does allow for impatience (i.e. $\delta < 1$)

- Lots of evidence of short-run impatience and long-run patience
  - which implies many individuals would choose (a) for question 1 and (b) for question 2

Frederick et al. (2002): Estimated δ increases by time horizon

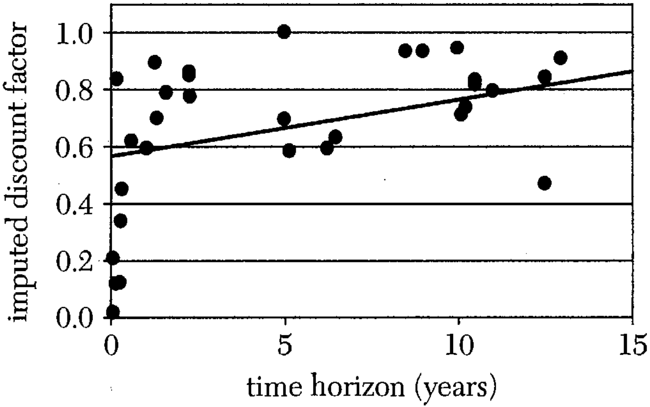

Figure: Frederick et al. (2002), Figure 1a

Evidence against the Exponential Discounting Model

- Short-run impatience and long-run patience
- **Time inconsistency**
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Time Consistency

- Time consistency (or dynamic consistency) = the action a person thinks she should take in the future always coincides with the action that she actually prefers to take once the time comes

- Time consistency an implication of the exponential discounting model
  - Consider the choice between two actions in period 1, A and B
  - At time $t = 0$, the individual prefers action A over B if and only if

$$u_0 + \delta u_1(A) + \delta^2 u_2(A) + \ldots \geq u_0 + \delta u_1(B) + \delta^2 u_2(B) + \ldots$$

  - Subtracting u_0 and dividing by δ gives

$$u_1(A) + \delta u_2(A) + \ldots \geq u_1(B) + \delta u_2(B) + \ldots$$

    which means the individual prefers A over B at time $t = 1$
  - That is, in the exponential discounting model, preferring A over B at $t = 0$ implies the individual will choose A over B at $t = 1$
    - i.e. the individual is time consistent

- Is time consistency realistic? Can you think of examples of time inconsistency?

Evidence against the Exponential Discounting Model

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Demand for Commitment

- Commitment device = a choice an individual makes in the present which restricts his set of choices in the future
- In the exponential discounting model, would the individual want a commitment device?
    - No. In this model, choices are time consistent so the future self will make whatever decision the present self prefers, whether or not choices are restricted.
- Can you think of examples of people demanding commitment devices?

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Quasi-Hyperbolic Discounting Model

At time t, the person aims to maximize

$$u_t + \beta\delta u_{t+1} + \beta\delta^2 u_{t+2} + \beta\delta^3 u_{t+3} + \ldots,$$

- What's the key difference between this model and the exponential discounting model?
  - β, the short-term discount factor
  - β relaxes the assumption that the amount of patience between any two periods is the same; it allows for more impatience between today and tomorrow than between 7 and 8 days from now
- Why is the quasi-hyperbolic discounting model a better fit, at least in some situations?
  - Its two parameters allow for short-run impatience and long-run patience
  - It predicts time-inconsistent behavior and demand for commitment

Quasi-hyperbolic discounting

Algorithm

Utility is given for each t by

$$U_t = \delta^{t-1} u_t(x_t) + \beta \sum_{s \geq t}^{T} \delta^{s-1} u_s(x_s). \tag{1}$$

The algorithm to solve the optimal plan $(x_t^*)_{t=1}^T$ is by **backwards induction**.

1. Determine $x_T^*(\cdot)$, a function of $(x_s)_{s<T}$.
    - first, calculate payoffs for each possible choice of x_T, given $(x_s)_{s<T}$
    - second, choose the best choice; this is the function x_T^*
2. Then use this information to determine $x_{T-1}^*(\cdot)$, as function of $(x_s)_{s<T-1}$.
3. Continue until you reach $t = 1$.

Example

Actions $x_t \in \{0, 1\}$. Payoffs

$$u_t(x) = \begin{cases} 0 & \text{if } x_t = 0 \text{ and } t < T \\ -\theta_t & \text{if } x_t = 1 \\ -\infty & \text{if } t = T \text{ and } x_s = 0 \text{ for all } s \leq T. \end{cases} \tag{2}$$

I.e., at T, if you have not done x_t, you must do it!

At T, optimal policy is $x_T^*(x) = 0$ if $x_t > 0$ for any $t < T$, and 1 otherwise.

At $T - 1$, it is more interesting. If $x_t = 0$ for all $t < T - 1$, then the optimal x_{T-1}^* is to delay to T if and only if

$$\theta_{T-1} > \beta\delta\theta_T.$$

$\therefore$ incentives to delay increase as $\beta \to 0$. ✓

Three-period example ($T = 3$)

Backwards induction:

- If $x_1 = x_2 = 0$, then $x_3^* = 1$.
- If $x_1 = 0$, then $x_2^* = 1 \iff -\theta_2 > -\beta\delta\theta_3$.
- Then payoffs from x_1 are

$$\begin{cases} -\theta_1 & \text{if } x_1 = 1 \\ -\beta\left[\delta\theta_2 x_2^* + \delta^2\theta_3 x_3^*\right] & \text{if } x_1 = 0 \end{cases} \tag{3}$$

  so that $x_1^* = 1 \iff -\theta_1 > \beta\left[\delta\theta_2 x_2^* + \delta^2\theta_3 x_3^*\right]$.

- So we deduce that

$$x^* = \begin{cases} (0,0,1) & \text{if } \beta\delta\theta_3 < \theta_2 \text{ and } \beta\delta^2\theta_3 < \theta_1 \\ (0,1,0) & \text{if } \theta_2 < \beta\delta\theta_3 \text{ and } \beta\delta\theta_2 < \theta_1 \\ (1,0,0) & \text{otherwise.} \end{cases} \tag{4}$$

  E.g., as $\beta \to 0$, $x^* = (0,0,1)$.
  As $\beta, \delta \to 1$, then $x^* = (1,0,0)$ (when θ_t increases in t).

Three-period example, continued

The x^* is the optimal policy or the agent's **behavior**. ✓

Welfare (utility) at $t = 1$ is given by

$$u(\theta, \delta, \beta) = \begin{cases} -\beta\delta^2\theta_3 & \text{if } \beta\delta\theta_3 < \theta_2 \text{ and } \beta\delta^2\theta_3 < \theta_1 \\ -\beta\delta\theta_2 & \text{if } \theta_2 < \beta\delta\theta_3 \text{ and } \beta\delta\theta_2 < \theta_1 \\ -\theta_1 & \text{otherwise.} \end{cases} \tag{5}$$

Now suppose the parameters are such that $x_1^* = 0$.

Demand for commitment. At $t = 1$, would prefer to commit to $x_2 = 1$ if

$$\theta_2 < \delta\theta_3$$

but in reality, will **not** do $x_2 = 1$ at $t = 2$ unless

$$\theta_2 < \beta\delta\theta_3.$$

Hence commitment has value when $\theta_2 \in [\beta\delta\theta_3, \delta\theta_3]$. In this region, the willingness to pay for a commitment device at $t = 1$ is $-\beta\delta(\theta_2 - \delta\theta_3)$.

Numerical example

Let $(\theta_1, \theta_2, \theta_3) = (\frac{8}{9}, 1, 2)$.

Let $\delta = 0.9$ and $\beta = \frac{1}{2}$. Recall the optimal policy is

$$x^* = \begin{cases} (0,0,1) & \text{if } \beta\delta\theta_3 < \theta_2 \text{ and } \beta\delta^2\theta_3 < \theta_1 \\ (0,1,0) & \text{if } \theta_2 < \beta\delta\theta_3 \text{ and } \beta\delta\theta_2 < \theta_1 \\ (1,0,0) & \text{otherwise.} \end{cases} \tag{4}$$

Check:

- $\beta\delta\theta_3 = \frac{1}{2} \cdot \frac{9}{10} \cdot 2 < 1 = \theta_2$ ✓
- $\beta\delta^2\theta_3 = \frac{1}{2} \cdot \frac{81}{100} \cdot 2 < \frac{8}{9} = \theta$ ✓

$\therefore$ Agent does the action at $t = 3$ by equation (4).

Would the agent prefer to do it at $t = 2$, from the viewpoint of $t = 1$? I.e., check if $\theta_2 < \delta\theta_3$:

$$\theta_2 = 1 < \delta\theta_3 = \frac{9}{10} \cdot 2$$

Indeed! The agent would. And the value of the commitment device is
$-\beta\delta(\theta_2 - \delta\theta_3) = -\frac{1}{2} \cdot \frac{9}{10} \cdot (1 - \frac{18}{10}) = \frac{1}{2} \cdot \frac{9}{10} \cdot \frac{4}{5} = \frac{9}{25}$.

Beliefs

Studying the model further.

Now, although the agent's true preferences are still given by (1) in each t, the agent **thinks** that it will behave in the future as if its β were some $\hat{\beta}$. Say

- "naïve" if $\hat{\beta} = 1$
- "sophisticated" if $\hat{\beta} = \beta$

This affects the calculation of the x^*'s, which depend on $\hat{\beta}$! In the example, use $\hat{\beta}$ in (4) rather than the true β.

Remark. Frank's shortcut. If $\hat{\beta} = 1$, then you can calculate all of the x^*'s as in a "standard" (exponential-discounting) dynamic optimization problem starting at each t.

But to evaluate payoffs, still use the true β. E.g.,

$$\tilde{u}(\theta, \delta, \beta, \hat{\beta}) = \begin{cases} -\beta\delta^2\theta_3 & \text{if } \hat{\beta}\delta\theta_3 < \theta_2 \text{ and } \hat{\beta}\delta^2\theta_3 < \theta_1 \\ -\beta\delta\theta_2 & \text{if } \theta_2 < \hat{\beta}\delta\theta_3 \text{ and } \hat{\beta}\delta\theta_2 < \theta_1 \\ -\theta_1 & \text{otherwise.} \end{cases} \quad (6)$$