Economics of Networks
Traffic Flow and Congestion Games

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Agenda

Traffic Routing and Equilibrium

Congestion Games

Reading: EK Chapter 8; Jackson Chapter 6

Physical Networks

Many interactions are constrained by physical networks

Roads and bridges

- Affect transit routing and congestion

Fiber-optic cables

- Ditto
- Cost-sharing

Geographic and political borders

- Affect trade, alliances, and conflicts

Strategic Traffic Routing

How to get to work in the morning

- What is the shortest route?
- Where are other people driving?

Optimal path depends on others' behavior

- This is a game

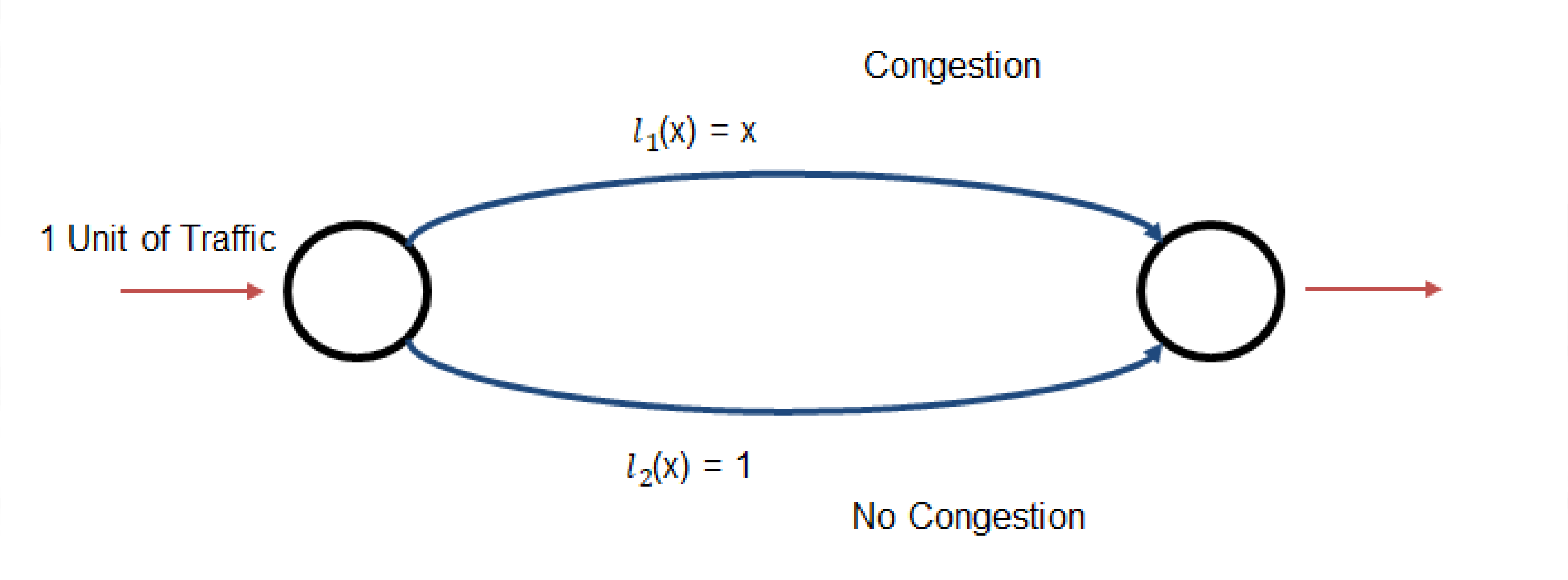
Questions:

- What do traffic patterns look like in equilibrium?
- How does equilibrium routing compare to optimal routing?

A Simple Example

Two routes from A to B, one unit of traffic

- Route 1, longer but more bandwidth
- Route 2, short but suffers congestion



Fixed latency $l_1(x) = 1$ on long route, latency $l_2(x) = x$ on short route

A Simple Example: Optimal Routing

One way to measure welfare: average latency

Traffic x_i on route i suffers delay $l_i(x_i)$

Seek to minimize

$$\sum_i x_i l_i(x_i)$$

Optimal routing splits traffic equally, giving

$$\frac{1}{2} l_1\left(\frac{1}{2}\right) + \frac{1}{2} l_2\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

A Simple Example: Equilibrium Routing

Suppose the unit mass of traffic comprises a continuum of infinitesimal players

- Individual deviations don't affect aggregate congestion

For any $x_2 < 1$, we have $l_2(x_2) < 1 = l_1(1 - x_2)$, so the marginal player always chooses route 2

Aggregate delay in equilibrium is

$$l_1(0) \cdot 0 + l_2(1) \cdot 1 = 0 + 1 = 1 > \frac{3}{4}$$

Inefficiency in equilibrium

The Social Cost of Traffic

Why are traffic patterns inefficient in equilibrium?

Key economic concept: externalities

Driving on a road imposes costs both on the person driving and on others

- Individual optimization fails to account for the effect of a decision on others' welfare

Can potentially get better outcomes by closing roads (example later) or by imposing tolls (see homework)

Generalizing the Model

For now, assume one origin-destination pair

- Route one unit of traffic

Directed network $N = (V, E)$

- Origin vertex o, destination vertex d

Set of paths $\mathcal{P}$ from origin to destination

- A path p is a collection of edges $i \in E$
- Flow x_p on path $p \in \mathcal{P}$

Generalizing the Model, continued

Each edge $i \in E$ handles traffic

$$x_i = \sum_{\{p \in \mathcal{P} \,:\, i \in p\}} x_p$$

Latency function $l_i(x_i)$

- Captures congestion
- Assume l_i nonnegative, differentiable, nondecreasing

Routing pattern is a nonnegative vector $\mathbf{x}$, elements sum to 1

- Flow over each possible path

Total latency (cost) of $\mathbf{x}$ is

$$C(\mathbf{x}) = \sum_{i \in E} x_i l_i(x_i)$$

Socially Optimal Routing

Benchmark: routing pattern that minimizes total cost

Routing pattern $\mathbf{x}^S$ that solves

$$\begin{aligned} \min \quad & \sum_{i \in E} x_i l_i(x_i) \\ s.t. \quad & \sum_{p \in \mathcal{P} \,:\, i \in p} x_p = x_i \quad \forall\, i \in E \\ & \sum_{p \in \mathcal{P}} x_p = 1 \\ & x_p \geq 0 \quad \forall\, p \in \mathcal{P} \end{aligned}$$

Equilibrium

In equilibrium, every motorist chooses best path given what others are doing

If a motorist chooses path p, there cannot exist a path p' such that

$$\sum_{i \in p'} l_i(x_i) < \sum_{i \in p} l_i(x_i)$$

Equilibrium conditions: there exists λ such that

- For any path p, we have $\sum_{i \in p} l_i(x_i) \geq \lambda$
- If $x_p > 0$, then $\sum_{i \in p} l_i(x_i) = \lambda$

Equilibrium Characterization

Theorem

A feasible routing pattern $\mathbf{x}^E$ is an equilibrium if and only if it solves

$$
\begin{aligned}
\min \quad & \sum_{i \in E} \int_0^{x_i} l_i(z) \mathrm{d}z \\
s.t. \quad & \sum_{\{p \in \mathcal{P} \,:\, i \in p\}} x_p = x_i \quad \forall\, i \in E \\
& \sum_{p \in \mathcal{P}} x_p = 1, \ \textit{and } x_p \geq 0 \quad \forall\, p \in \mathcal{P}
\end{aligned}
$$

If each l_i is strictly increasing, then $\mathbf{x}^E$ is unique.

Note by Weierstrass's Theorem a solution exists, so an equilibrium exists

Proof

Rewrite the minimization problem as

$$\begin{aligned} \min \quad & \sum_{i \in E} \int_0^{\sum_{i \in p} x_p} l_i(z) \mathrm{d}z \\ s.t. \quad & \sum_{p \in \mathcal{P}} x_p = 1, \text{ and} x_p \geq 0 \quad \forall p \in \mathcal{P} \end{aligned}$$

Lagrangian

$$\sum_{i \in E} \int_0^{\sum_{i \in p} x_p} l_i(z) \mathrm{d}z - \lambda \left(\sum_{p \in \mathcal{P}} x_p - 1 \right) - \sum_{p \in \mathcal{P}} \mu_p x_p$$

Convex problem, FOC is necessary and sufficient

- FOC for x_p is
$$\sum_{i \in p} l_i(x_i^E) = \lambda + \mu_i$$
- Complementary slackness: $\mu_i \geq 0$ with equality if $x_p > 0$

Proof, continued

If $x_p = 0$, FOC implies $\sum_{i \in p} l_i(x_i^E) = \lambda + \mu_p \geq \lambda$

- Recall our first equilibrium condition

If $x_p > 0$, FOC implies $\sum_{i \in p} l_i(x_i^E) = \lambda$

- Recall our second equilibrium condition

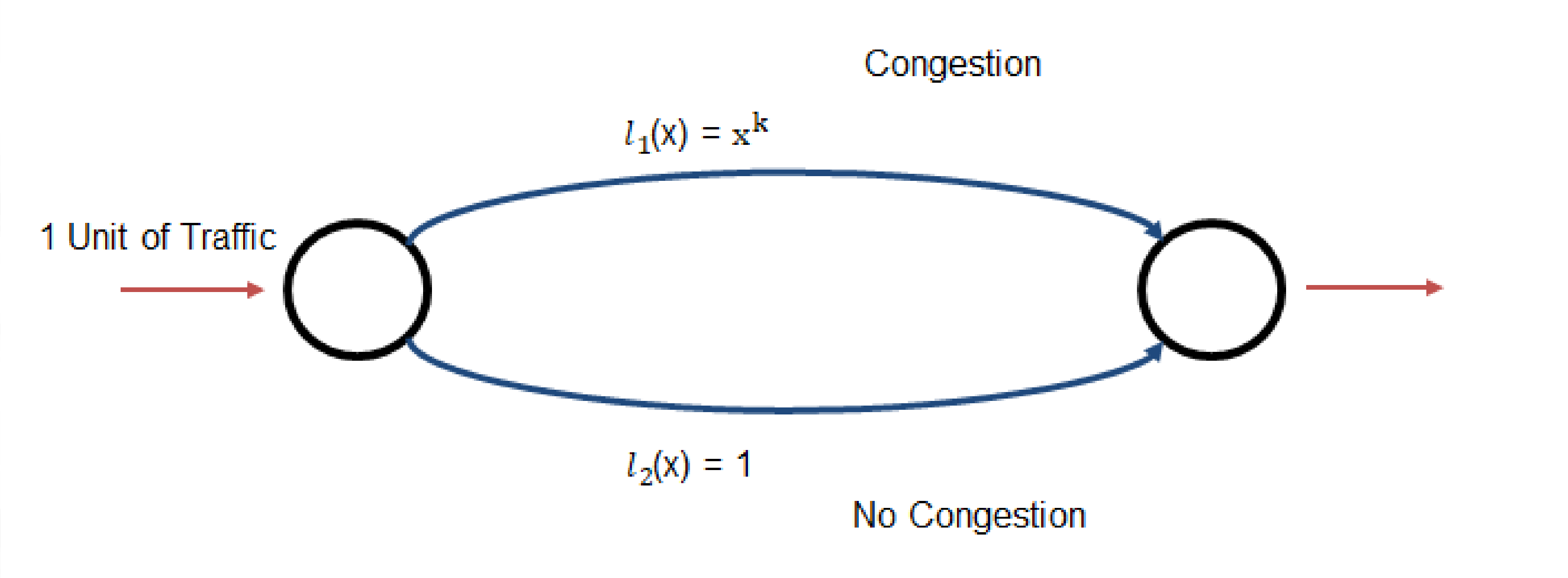
Uniqueness when each l_i is strictly increasing...left as an exercise

Extent of Inefficiency

Recall our simple example showing that equilibirum fails to minimize total cost

- Equilibrium can be inefficient

Equilibrium can be arbitrarily inefficient



Extent of Inefficiency, continued

Socially optimal routing solves

$$\begin{aligned} \min \quad & x_1 + x_2^{k+1} \\ s.t. \quad & x_1 + x_2 = 1, \quad x_1, x_2 \geq 0 \end{aligned}$$

First order conditions imply

$$(k+1)x_2^k = 1 \quad \implies \quad x_2 = (k+1)^{-\frac{1}{k}}$$

Total cost is then

$$C(\mathbf{x}^S) = 1 - (k+1)^{-\frac{1}{k}} + (k+1)^{-\frac{k+1}{k}}$$

which approaches 0 as $k \to \infty$

Extent of Inefficiency, continued

In equilibrium, we again have $x_1 = 0$ and $x_2 = 1$

- Same argument as before

Total cost in equilibrium is

$$C(\mathbf{x}^E) = 0 + 1 = 1$$

independent of k

Ratio $\frac{C(\mathbf{x}^S)}{C(\mathbf{x}^E)}$ tends to zero with k

- Equilibrium can be arbitrarily inefficient relative to optimum

Braess's Paradox

Additional routes can negatively impact network users

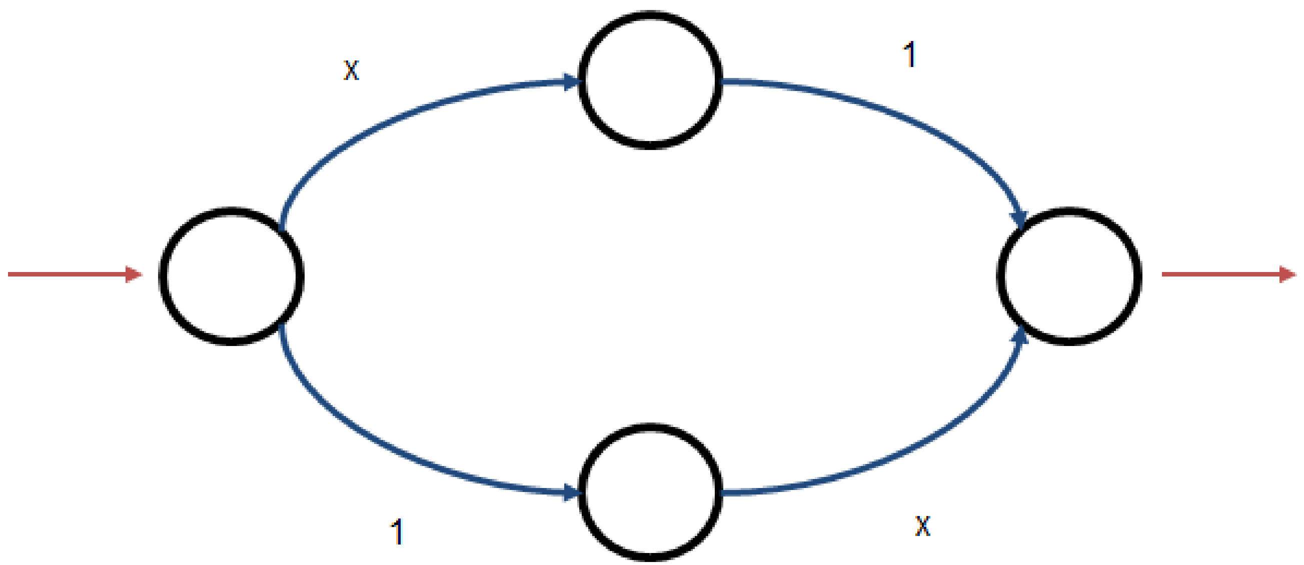
Parodox because more routes should only help traffic

- Could always leave routing unchanged
- Social optimum can only get better

Equilibrium response can change this

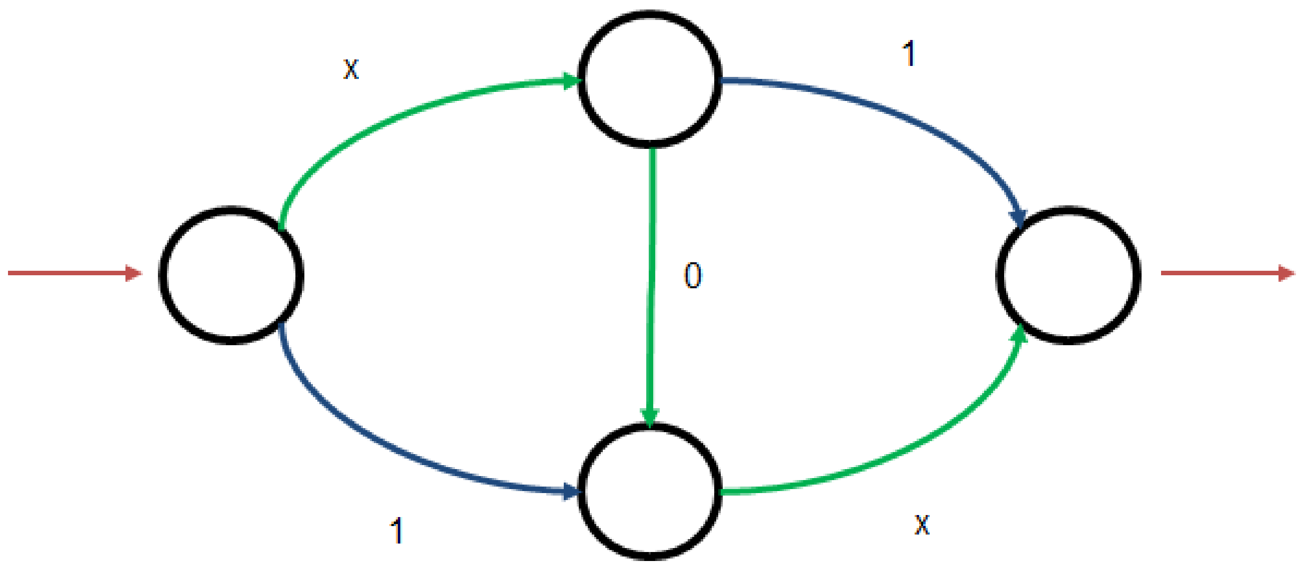
- Expalins why closing a road might improve traffic in congested city

Braess's Paradox



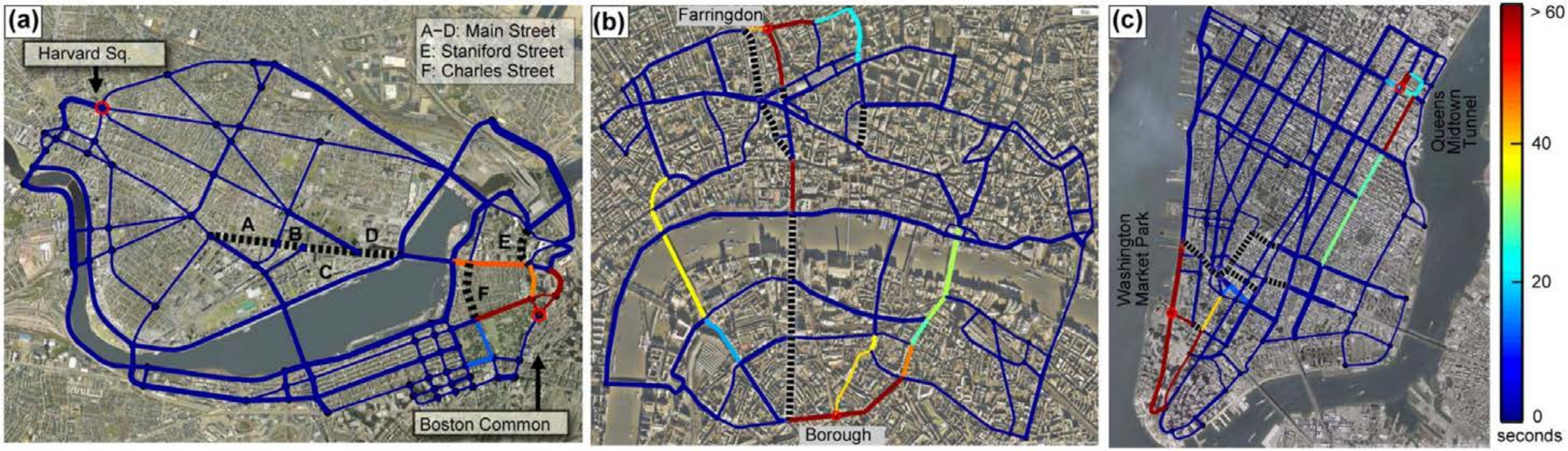
In equilibrium, flow $\frac{1}{2}$ on each route, average cost $\frac{3}{2}$

Braess's Paradox



In equilibrium, all take highlighted route, average cost 2

Braess's Paradox



Studies suggest that closing streets in black will reduce congestion.

- (See Youn et al., "Price of Anarchy in Transportation Networks: Efficiency and Optimality Control")

Congestion Games

Traffic routing game belongs to larger class of congestion games

- Internet traffic
- Airport traffic
- Supermarket checkout lines

Previously assumed players were "small"

In some cases, one player's action can significantly impact congestion

- e.g. Delta using Atlanta as a hub has a material impact on airport congestion in Atlanta

Congestion Games

Congestion model $C = \{N, R, (S_i)_{i\in N}, (c^j)_{j\in R}\}$

- Set of players $N = \{1, 2, ..., n\}$
- Set of resources $R = \{1, 2, ..., r\}$
- Resource combinations i can use $S_i \subset R$
- Benefit of resource j if k players use it $c^j(k)$ (possibly negative)

Congestion game $\{N, (S_i)_{i\in N}, (u_i)_{i\in N}\}$ with utilities

$$u_i(s_i, s_{-i}) = \sum_{j\in s_i} c^j(k_j)$$

Congestion games have a useful structure...

Potential Games

A game is a potential game if there exists a potential function $\Phi : S \to \mathbb{R}$ that characterizes players' payoffs

Ordinal potential function if for all $s_{-i} \in S_{-i}$ and all $x, z \in S_i$:

- $u_i(x, s_{-i}) - u_i(z, s_{-i}) \geq 0$ i $\Phi(x, s_{-i}) - \Phi(z, s_{-i}) \geq 0$
- $u_i(x, s_{-i}) - u_i(z, s_{-i}) > 0$ i $\Phi(x, s_{-i}) - \Phi(z, s_{-i}) > 0$

Exact potential function if for all $s_{-i} \in S_{-i}$ and all $x, z \in S_i$:

- $u_i(x, s_{-i}) - u_i(z, s_{-i}) = \Phi(x, s_{-i}) - \Phi(z, s_{-i})$

For each player, a best response maximizes the potential function given others' strategies

Pure Strategy Equilibria in Potential Games

A game G is an ordinal (exact) potential game if it admits an ordinal (exact) potential function

Theorem

If G is a potential game with S finite or compact, then G has at least one pure strategy Nash equilibrim

Proof: The global maximum of the potential function corresponds to a pure strategy Nash equilibrium

Note: result says nothing about uniqueness

Example of Ordinal Potential Game

Example: Cournot competition

- Each of N firms chooses quantity q_i, define $Q = \sum_{i=1}^{N} q_i$
- Payo for firm i is $u_i(q_i, q_{-i}) = q_i(P(Q) - c)$

Define the function

$$\Phi(q_1, \cdots, q_N) = \left(\prod_{i=1}^{N} q_i\right)(P(Q) - c)$$

For all i and all q_{-i}, we have $u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) > 0$ i $\Phi(q_i, q_{-i}) - \Phi(q_i', q_{-i}) > 0$ for all $q_i, q_i' > 0$

Φ is an ordinal potential function for this game

Example of Exact Potential Game

Example: Cournot competition (again)

- Each of N firms chooses quantity q_i, define $Q = \sum_{i=1}^{N} q_i$
- Assume linear demand, payo $q_i \left(a - b(Q) - c\right)$

Define the function

$$\Phi(q_1, \cdots, q_N) = \left(\sum_{i=1}^{N} aq_i - bq_i^2\right) - b \sum_{1 \leq i \leq j \leq N} q_i q_j$$

Exercise: show that

$$u_i(q_i, q_{-i}) - u_i(q_i', q_{-i}) = \Phi(q_i, q_{-i}) - \Phi(q_i', q_{-i})$$

for all $q_i, q_i' > 0$

Congestion Games as Potential Games

Theorem

Every congestion game is a potential game and therefore has a pure strategy equilibrium

Proof: Fix the strategy profile $\mathbf{s}$. For each resource j, let $\overline{k}_j^i$ denote the number of users of j excluding player i

The utility difference for player i between s_i and s'_i is then

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = \sum_{j \in s_i} c^j(\overline{k}_j^i + 1) - \sum_{j \in s'_i} c^j(\overline{k}_j^i + 1)$$

Congestion Games as Potential Games

Given a profile s, let $J_{\mathbf{s}}$ denote the resources used by at least one player, and let $J_{s_{-i}}$ denote the resources used by at least one player excluding i

Consider the function

$$\Phi(\mathbf{s}) = \sum_{j \in J_{\mathbf{s}}} \left[\sum_{k=1}^{k_j} c^j(k) \right]$$

Which we can rewrite as

$$\Phi(s_i, s_{-i}) = \sum_{j \in J_{s_{-i}}} \left[\sum_{k=1}^{\overline{k}_j^i} c^j(k) \right] + \sum_{j \in s_i} c^j(\overline{k}_j^i + 1)$$

Congestion Games as Potential Games

Therefore:

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = \sum_{j \in J_{s_{-i}}} \left[\sum_{k=1}^{\overline{k}_j^i} c^j(k) \right] + \sum_{j \in s_i} c^j(\overline{k}_j^i + 1)$$

$$- \sum_{j \in J_{s_{-i}}} \left[\sum_{k=1}^{\overline{k}_j^i} c^j(k) \right] + \sum_{j \in s_i'} c^j(\overline{k}_j^i + 1)$$

$$= \sum_{j \in s_i} c^j(\overline{k}_j^i + 1) - \sum_{j \in s_i'} c^j(\overline{k}_j^i + 1)$$

$$= u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})$$

Takeaways

Physical networks constrain many interactions, and the structure of these networks therefore guides individual decisions

Strategic interactions have some non-obvious implications

- Equilibrium behavior can be much worse than the social optimum
- Closing roads can make everyone better off

Next time: more externalities and network effects

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