14.271: Problem Set 1 Solutions

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1. A monopolist can produce different versions of a good with different quality levels. The constant marginal cost of producing quality v goods is cv^2 with $c < 1$. The monopolist is selling to a continuum of consumers with types θ distributed uniformly on $[0, 1]$. The types reflect vertical heterogeneity in tastes: a type θ consumer gets utility $\theta v - p$ if she buys one unit of a quality v good at price p and zero utility if she does not buy. (Assume that consumers always buy at most one unit.)

   (a) Suppose first that the monopolist can only produce a single quality level v. Solve for the monopolist's price and its profits as a function of c and v.

   For some fixed v, choosing p is equivalent to choosing the threshold type $\widehat{\theta}$ that buys. This type is defined by

$$\widehat{\theta} = \frac{p}{v}$$

   and for $\theta \sim \mathcal{U}[0, 1]$ the total mass that buys is $(1 - \widehat{\theta})$. The monopolist's problem is then

$$\max_{\widehat{\theta} \leq 1} \quad (1 - \widehat{\theta})(v\widehat{\theta} - cv^2) \tag{1}$$

   so the optimum is at

$$-(v\widehat{\theta} - cv^2) + v(1 - \widehat{\theta}) = 0 \implies \widehat{\theta}^*(v) = \min\left\{\frac{1 + cv}{2},\ 1\right\} \tag{FOC$_\theta$}$$

   and the second order condition is satisfied. Plugging back in, prices and profits are, as functions of v,

$$p(v) = \min\left\{\frac{v(1 + cv)}{2},\ v\right\}$$

$$\pi(v) = \frac{v(1 - cv)^2}{4} \qquad \text{if } cv \leq 1,\ = 0 \text{ otherwise} \tag{2}$$

*based on solutions by Adam Harris, Anton Popov, and Sam Grondahl

(b) What quality level would the firm choose in the model of part (a) if v were a choice variable?

Maximizing (2) with respect to v gives

$$(1-cv)^2 - 2cv(1-cv) = 0 \implies v^* = \frac{1}{3c} \qquad (\text{FOC}_v)$$

Above we required that $\widehat{\theta} \in [0,1]$. For the optimal v^* we have $cv < 1$ and $\widehat{\theta}^* = \frac{2}{3}$ is interior.

(c) What quality level would a social planner choose if the social planner had the ability to choose both v and p? Discuss how this compares with the outcome of part (b) and how we can think about this outcome in light of standard results on a monopolist's choice of product quality.

Total welfare – the planner's maximand – is

$$\int_{\widehat{\theta}}^{1} (v\theta - cv^2)d\theta = \frac{v(1-\widehat{\theta}^2)}{2} - (1-\widehat{\theta})cv^2 \qquad (3)$$

We can rewrite the monopolist's maximand (1) equivalently as

$$\int_{\widehat{\theta}}^{1} (v\widehat{\theta} - cv^2)d\theta \qquad (4)$$

Comparing (3) with (4) we can see that, if they choose the same level of $\widehat{\theta}$, the monopolist internalizes the effect on the *marginal* consumer whereas the planner internalizes the effect on the *average* (purchasing) consumer, which would lead the monopolist to choose a lower level of v. However, the monopolist will not choose the same level of $\widehat{\theta}$ as the planner. To find the planner's solution, we take the same approach as in (a) and (b). Maximizing (3) over $v, \widehat{\theta}$ gives

$$\widehat{\theta}^P = cv^P$$
$$v^P = \frac{1+\widehat{\theta}^P}{4c}$$

Combining the FOCs gives the optima

$$v^P = \frac{1}{3c}\ ; \quad \widehat{\theta}^P = \frac{1}{3}$$

Comparing this outcome to that in part (b) we can see that there is no quality distortion: $v^* = v^P$. On the other hand, plugging v^* back into the expression for $\widehat{\theta}^*$ in (a) gives $\widehat{\theta}^* = \frac{2}{3} > \widehat{\theta}^P$, so the monopolist restricts output relative to the planner. In this particular example, the output restriction happens to be such that the (marginal) utility of quality to monopolist's *marginal* consumer exactly equals that to the planner's *average* consumer, leading them to choose the same level of v, but this is not a general property.

2. Consider a two-period durable good model. As in class, assume for simplicity that there is no discounting by either the firm or consumers. There is a continuum of consumers of unit mass with types $\theta \sim \mathcal{U}[0,1]$. In the first period, the monopolist is able to produce a durable good of quality s_1 at cost c. In the second period there is technological progress, so the monopolist can produce a good of quality $s_2 > s_1$ at cost c. A type θ consumer gets utility $\theta s - p$ in any period in which she consumes a good of quality s and pays p. Suppose that an efficient resale market exists at $t = 2$ so that high θ consumers can resell their used goods to lower θ consumers if they want to purchase a new good. Assume that $s_1 > \frac{c}{2}$ and $s_2 > c$ (so that a type $\theta = 1$ consumer values both goods more than their production cost).

   (a) What prices will the monopolist need to charge in equilibrium in the two periods if he wants to sell q_1 units at $t = 1$ and q_2 units at $t = 2$? What will be the price in the resale market for used goods in this case?

   Let p_t denote the price of a new good in period $t = 1, 2$, and p^u denote the price of a used good in period 2. A small trick to make the problem easier is to think of efficient resale market in the following way: All consumers who bought the good in period 1 sell it at the end of period 1 to the middle man at price p^u. At the start of period 2 the middle man sells all those goods (q_1) at price p^u to consumers who want to have a used good in period 2 (among those may be the consumers who owned the good in period 1, and those who did not).

   At $t = 2$ we can think of three groups of consumer types according to what they will buy: those highest types who buy the new good, those medium types who buy the old good in the resale market, and those lowest types who do not buy any good. Specifically, we can partition the space into three regions with cutoffs $\underline{\theta} \leq \bar{\theta}$: types $[0, \underline{\theta}]$ buy nothing, types $[\underline{\theta}, \bar{\theta}]$ buy the used good, and types $[\bar{\theta}, 1]$ buy the new good.

   The partitioning must look like this because of increasing differences of utility in (s, θ) and the fact that $s_2 > s_1$. Suppose there exists a person with type $\tilde{\theta}$ who wants to buy a new good in period 2. It must hold for her that

$$s_2\tilde{\theta} - p_2 \geq s_1\tilde{\theta} - p^u$$

$$s_2\tilde{\theta} - p_2 \geq 0$$

   By increasing differences and $s_2 > s_1$, the same inequalities will hold for any $\theta > \tilde{\theta}$. So, if a positive quantity q_2 of goods is sold in period 2, it must be that types $\theta \in [1 - q_2, 1]$ buy those goods, and $\bar{\theta} = 1 - q_2$.

   Similarly, we may argue that the types who prefer buying the used good in period 2 are located to the right of the types who prefer not bying anything in period 2. There is a total quantity q_1 of used goods, and the resale market clears. So, it must be that types $\theta \in [1 - q_2 - q_1, 1 - q_2]$ buy the used good, $\underline{\theta} = 1 - q_2 - q_1$.

The lower threshold type $\underline{\theta}$ is indifferent between buying the old good and not buying, so

$$\underline{\theta}s_1 - p^u = 0 \tag{5}$$

$$p^u = s_1(1 - q_1 - q_2) \tag{6}$$

Threshold type $\bar{\theta}$ is indifferent between purchasing the old good and purchasing the new good, so

$$\bar{\theta}s_1 - p^u = \bar{\theta}s_2 - p_2 \Rightarrow p_2 = p^u + \bar{\theta}(s_2 - s_1) \tag{7}$$

$$p_2 = p^u + (1 - q_2)(s_2 - s_1) \tag{8}$$

Intuitively, the price the monopolist can charge for the new good in period 2 is the price of the used good plus the extra value of the new good over the old for the type $\bar{\theta}$ consumer. Combining (6) and (8), and simplifying gives

$$p_2 = s_2(1 - q_2) - s_1 q_1 \tag{9}$$

Now consider the first period. Because utility is increasing in θ, there is a single threshold $\hat{\theta}$ that separates those who buy and those who do not. Since there are q_1 units available in period 1 and $1-\hat{\theta}$ consumers buy in period 1, we must have $1 - \hat{\theta} = q_1$, so

$$\hat{\theta} = 1 - q_1 \tag{10}$$

Type $\hat{\theta}$ is indifferent between buying today and not. Since he will sell the good at the end of period 1, this implies that

$$\hat{\theta}s_1 - p_1 + p^u = 0 \tag{11}$$

(The user cost for period 1 is $p_1 - p^u$.) Combining (6), (10) and (11) therefore gives

$$p_1 = 2s_1(1 - q_1) - s_1 q_2 \tag{12}$$

(b) Write down the monopolist's maximization problem and solve for the optimal pricing policy for a monopolist with commitment power assuming that the parameters are such that the monopolist wants to make sales in both periods. What conditions on s_1, s_2, and c are necessary for the solution you've found to be the true solution to the profit-maximization problem? How does this fit with what we saw about the solution to the durable goods problem in class?

The monopolist's maximand is $\Pi(q_1, q_2) = (p_1(q_1, q_2) - c) \cdot q_1 + (p_2(q_1, q_2) - c) \cdot q_2$. Substituting in (9) and (12), the maximand becomes

$$\Pi(q_1, q_2) = (2s_1(1 - q_1) - s_1 q_2 - c) \cdot q_1 + (s_2 - s_1 q_1 - s_2 q_2 - c) \cdot q_2 \tag{13}$$

Maximizing over both arguments and solving the system of FOCs gives

$$q_1^* = \frac{s_1 s_2 - c(s_2 - s_1)}{2s_1(2s_2 - s_1)}$$
$$q_2^* = \frac{2(s_2 - s_1) - c}{2(2s_2 - s_1)}$$

and the corresponding prices are

$$p_1^* = s_1 + \frac{c}{2}$$
$$p_2^* = \frac{s_2 + c}{2}$$

In equilibrium we require the following conditions:

($q_1^*, q_2^* \geq 0$) Since $s_2 > s_1$ and $s_1, s_2 > 0$ both denominators are strictly positive. To ensure nonnegativity on q_1^* it is therefore sufficient to have $s_1 s_2 \geq c\Delta s$. Similarly, the condition for $q_2^* \geq 0$ is $\Delta s \geq c/2$. We can combine these conditions to obtain the composite condition (which is necessary but not sufficient) that $s_1 s_2 \geq c^2/2$, which is guaranteed by the assumptions in the problem. However, we need additional conditions on the relative separation Δs as well, so the composite is not sufficient.

($q_1^* + q_2^* \leq 1$) In this model, as in the durable goods model in class, consumers only consume a single good at any time, so the total quantity of goods in the economy cannot exceed the mass of consumers. Adding the quantities gives

$$\frac{3s_1 s_2 - 2s_1^2 - cs_2}{s_1(4s_2 - 2s_1)} \leq 1$$

We can rewrite the numerator as $s_1(4s_2 - 2s_1) - (s_1 + c)s_2$ and the constraint becomes

$$1 - \frac{(s_1 + c)s_2}{s_1(4s_2 - 2s_1)} \leq 1$$

from which it is clear that the constraint is always satisfied for $s_1, s_2, c > 0$, $s_2 > s_1$.

($p_1, p_2 \geq c$) This condition, which ensures that the monopolist does not sell either good at a loss, is satisfied for $s_1 \geq c/2$ and $s_2 \geq c$, which is guaranteed by the assumption in the problem.

We can also check that the second order conditions hold, so the solution is a maximum.

In this model, subject to strict versions of the conditions above, we have positive sales in both periods whereas in the durable goods model presented in class we had $q_2 = 0$. The central difference is that quality improves in the second period in our model. Indeed, for $\Delta s \leq c/2$ we have no sales in the second period.

Setting $s_1 = s_2 = 1$, $c = 0$ gives us the solution we saw in class, with sales only in the first period at $p_1 = 1$, $q_1 = \frac{1}{2}$. In period 2 the middle man just sells used goods to the same $\frac{1}{2}$ of consumers who used to have them in the period 1.

As in class, the monopolist has commitment power in this model. Here he internalizes the effect of q_2 on the resale price p^u which in turn affects q_1. In the no-commitment model the monopolist will want to re-optimize in period 2. That will make him want to oversell in period 2 relative to the model with commitment.

3. Read at least the introduction and initial model description in Bergemann and Valimaki's 2006 *JPE* paper "Dynamic Pricing of New Experience Goods." Think about the paper in relation to Shapiro's 1983 *Bell Journal* paper "Optimal Pricing of Experience Goods." What shortcomings of Shapiro's paper do Bergemann and Valimaki try to address in their model? To what extent do their results reflect the main intuitions of Shapiro's analysis of what happens with "pessimistic" and "optimistic" beliefs? In what ways have they made special assumptions or taken steps backward from Shapiro's model in order to keep their analysis tractable?

   As this question was open-ended asked you to think in broad terms about two different models, there is no single correct answer. Below is one exemplary answer, written by Kelsey Moran and reproduced here with her permission:

   An experience good is defined as a product that consumers learn about through their own experience with it. When an experience good is first introduced to the market, all buyers are uncertain about the quality of the good; only by consuming the good are buyers able to the true quality. Thus, we can say that information about experience goods is bundled with the product itself. Any model involving experienced goods should hence be a dynamic one.

   Shapiro (1983) is one of the first papers to develop such a model, investigating the profit-maximizing dynamic pricing plan for the seller of an experience good. In Shapiro's model, a consumer of type θ receives utility $\theta q - p$ from purchasing one unit of the experience good, where q denotes quality and p denotes price. Consumers are initially uninformed about the true quality of the good, but they have some belief about what the true quality is, designated by R. Once a consumer purchases the good, however, he immediately learns the true q (with probability 1), and this q does not change over time. Consumers thus make the discrete choice to either buy or not buy the good in each period. Restricting the analysis to a set of consumers with the same initial expectations for product quality, Shapiro splits this problem into two cases: (1) the optimistic case, where consumers initially overestimate product quality ($R > q$), and (2) the pessimistic case, where consumers initially underestimate product quality ($R < q$). In the pessimistic case, Shapiro finds that the optimal pricing policy is two-staged: a low price in the first period—to draw in consumers—followed by a constant higher price. In the optimistic case, Shapiro finds that the optimal pricing policy is more complex so that the seller can milk its initial reputation to the fullest: specifically, a declining price path followed by a jump up to a terminal price.

Bergemann and Valimaki (2006; now referred to as BV) highlight two shortcomings from simplifying assumptions in the model used by Shapiro (1983; now referred to as S):

- Each consumer acts myopically;
- The expectation of the quality is biased with respect to the true expected equality.

As the authors state, "the myopic consumer fails to evaluate the option value of the experiment, and with the natural benchmark of expectationally unbiased buyers, his (Shapiro's) model predicts constant prices." To address the first of these shortcomings, BV allow for forward-looking consumers who discount at rate $r > 0$. Similar to S, consumers in the BV model are indexed by their types, θ , which—in the BV model—represent their idiosyncratic willingness to pay rather than their tastes for product quality. The BV consumers are thus learning about their types (rather than learning about the quality of the good, as in S)—i.e., each consumer's true value of θ is initially unknown to both the buyer and the seller, and only once a consumer purchases the experience good does he learn his true type (at constant Poisson rate λ, which results in the state variable of the BV model being the fraction of informed buyers in each period). Further, note that BV assume that all buyers are ex ante identical. This complete homogeneity of consumers ex ante could be considered a step back from the S model, where consumers had homogenous initial expectations of product quality but heterogenous tastes for quality.

BV show that it is possible to classify all markets as either: (1) mass markets, in which consumers are willing to purchase at the optimal static monopoly price under full information or (2) niche markets, in which consumers are not willing to do so. In niche markets, uninformed buyers do not purchase at the static monopoly price, and thus BV find that the monopolist must offer low initial prices in order to motivate learning. In mass markets, in contrast, uninformed buyers purchase in all periods so BV find that the monopolist skims in the early stages the more attractive part of the market (i.e. the uninformed buyers) by using a dynamic equilibrium in which prices start high and slowly decrease over time. Thus, despite the modeling differences between S and BV, the qualitative results of both models are consistent with one another. The equilibrium strategy in the niche markets of BV mirrors that of the pessimistic case in S (low initial prices to motive learning), while the equilibrium strategy in the mass markets of BV mirrors that of the optimistic case in S (declining prices to milk/skim as much as possible).