14.271: PROBLEM SET 2

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1

Consider a two-type model of price discrimination without unit demands. A monopolist produces a divisible good at a constant marginal cost of zero. There is a unit mass of consumers. They may buy any nonnegative real number of units of the good. Half of the consumers are type $\theta = 1$. Each has inverse demand function $P_1(q) = 1 - q$. The other half of consumers are of type $\theta = 2$. Each has inverse demand function $P_2(q) = A - bq^2$ with $A > 0$ and $b > 0$.

(Note that there is no free disposal in this question. A consumer's utility is reduced if he or she is given more units than he or she wants.)

(a) Consider first the best situation from the monopolist's perspective: suppose that θ is observable and the monopolist can charge any tariff $T(q, \theta)$, i.e. the firm can use nonlinear prices and can set separate prices in the two populations with no worries about monitoring, arbitrage, etc.

Find an optimal pricing policy for the firm. What is the firm's profit?

The monopolist will solve

$$\Pi = \frac{1}{2} \max_{T_1, q_1, T_2, q_2} T_1 - cq_1 + T_2 - cq_2$$

where $c = 0$ is the marginal cost and the monopolist has to respect the IR constraints $\int_0^{q_\theta} P_\theta(x)\ dx \geq T_\theta$ for $\theta = 1, 2$.[2] (Essentially, the monopolist is solving a separate problem for each market.) Clearly, the IR constraint will be binding, and the monopolist will choose q_θ to be the root of P_θ and T_θ to be the gross surplus of consumer θ at this value of q_θ. Then, $1 - q_1^* = 0 \implies q_1^* = 1$ and thus $T_1^* = \int_0^1 P_1(x)\ dx = 1 - 1/2 = 1/2$. Similarly, $A - bq_2^* = 0 \implies q_2^* = \sqrt{A/b}$. Thus, $T_2^* = \int_0^{\sqrt{A/b}} (A - bx^2)\ dx = \frac{2}{3} A\sqrt{A/b}$.

Note that there are a number of ways to implement this first-best. The monopolist could charge $T(q, \theta) = \int_0^q P_\theta(x)\ dx$, and the consumer would be indifferent between consuming anywhere between 0 and the q_θ^* determined above. (We would assume that the consumer buys the largest amount of

[1] based on solutions by Adam Harris, Anton Popov, Vivek Bhattacharya, and Sam Grondahl

[2] The 1/2 comes from the fact that there is a mass of 1/2 of each type of consumer. This is not the important part of the problem.

the good possible, or incentivize him by reducing $T(q_\theta^*, \theta)$ by an arbitrarily small ϵ.) The monopolist could also charge a forcing contract:

$$T(q,\theta) = \begin{cases} T(q_\theta^*, \theta) \text{ if } q = q_\theta^*, \\ 0 \text{ otherwise} \end{cases}$$

The profit of the monopolist is

$$\Pi = \frac{1}{2}\left[T_1^* + T_2^*\right] = \frac{1}{2}\left[\frac{1}{2} + \frac{2}{3}\sqrt{\frac{A^3}{b}}\right].$$

(b) Are there values for (A, b) for which the monopolist would be able to receive the same profit as in part (a) even if the monopolist could not observe θ, i.e. if the monopolist were restricted to using a tariff of the form $T(q)$?

Prove that this is never possible or provide a set of parameter values for which you can show that a tariff of the form $T(q)$ suffices.

We essentially add the IC constraints. As long as $\int_0^{q_1^*}(A - bx^2)\,dx - T_1^* \leq 0$ and $\int_0^{q_2^*}(1-x)\,dx - T_2^* \leq 0$, we can ensure that type 1 consumers do not buy the type 2 bundle, and vice versa. This amounts to $A - b/3 - 1/2 \leq 0$ and $1 - (\sqrt{A/b})/2 - 2A/3 \leq 0$, which is a nonempty region.

I can present an extended example (in which $T(q)$ is nontrivial for all q as well, although that's certainly not necessary). Consider $A = 1$ and $b = 2$. Consider the tariff schedule

$$T(q) = \begin{cases} q - 2q^3/3 & \text{if } q \leq \sqrt{1/2} \\ q - q^2/2 & \text{if } q > \sqrt{1/2} \end{cases}.$$

Note that a type 2 consumer would not buy more than $q = 1/\sqrt{2}$. A type 2 consumer earns a surplus of $\int_0^q P_2(x)\,dx - T(q) = 0$ for any $q \in [0, 1/\sqrt{2}]$ Now note that a type 1 consumer's surplus is $q - q^2/2 \leq q - 2q^3/3$ for $q \in [0, \sqrt{1/2}]$.[3] However, a type 1 consumer's surplus is $\int_0^q 1 - x\,dx - (q - q^2/2) = 0$ for $q \geq 1/\sqrt{2}$. Thus, a type 1 consumer is indifferent between buying $q \in [1/\sqrt{2}, 1]$. We arbitrarily break the indifference conditions by setting $q_2^* = 1/\sqrt{2}$ and $q_1^* = 1$, the maximum they would be willing to buy. This of course gives the monopolist the same profit as in the first best.[4]

[3] Really, this is true until $q = 3/4$, but a type 2 consumer would not be willing to buy more than $\sqrt{1/2}$, so we can switch schedules at $q = 1/\sqrt{2}$ and not worry about the discontinuity.

[4] In general, we need the surplus for the type 2 for $q < \sqrt{A/b}$ to be more than the the surplus for type 1. That is, we need $q - q^2/2 < Aq - bq^3/3$. It suffices to check that this inequality holds at $q = \sqrt{A/b}$.

(c) Suppose now that $A = 1$ and $b = \frac{1}{2}$. The type 2 consumers can now be thought of as "high types" who have at least a weakly higher valuation for each unit.

Suppose again that θ is unobservable. Suppose also that the monopolist can monitor which consumers are using the good, but cannot prevent resale among the consumers. Hence, the only feasible tariffs will be two part tariffs of the form $T(q) = C + pq$.

Show that the optimal policy for the monopolist will have $p > 0$. What about this situation is different from the textbook example of two-part tariffs where the monopolist sets $p = c$ and extracts all the surplus using a fixed fee?

Given a pricing schedule $C + pq$, the type θ consumer chooses q to solve

$$\max_q \int_0^q P_\theta(x)\ dx - C - pq$$

subject to the fact that this quantity is positive. A type 1 consumer then chooses q_1^* so that $1 - q_1^* - p = 0 \implies q_1^* = 1 - p$. Similarly, a type 2 consumer chooses $1 - q_2^*/2 - p = 0$, or $q_2^* = \sqrt{2(1-p)}$. Therefore, the surplus (net of the tariff) of the consumers is

$$S_1 = (1-p) - \frac{(1-p)^2}{2} - C - p(1-p) = \frac{(1-p)^2}{2} - C$$
$$S_2 = \sqrt{2(1-p)} - \frac{1}{6}\left(\sqrt{2(1-p)}\right)^3 - C - p\sqrt{2(1-p)}$$

The monopolist choose C and p to solve

$$\frac{1}{2}\max_{C,p} p\left(q_1^* + q_2^*\right) + 2C = \frac{1}{2}\max_{C,p} p\left((1-p) + \sqrt{2(1-p)}\right) + 2C$$

subject to the individual rationality constraints $S_\theta \geq 0$ for both θ. The only binding individual rationality constraint is that for the "low" type 1, so we have that $C = (1-p)^2/2$. Then, the maximand becomes $p\left((1-p) + \sqrt{2(1-p)}\right) + (1-p)^2$. This quantity has a root at $p = 1$ and is 1 at $p = 0$. But, the derivative of this expression is

$$-1 + \frac{\sqrt{2}}{\sqrt{1-p}} - \frac{3p}{\sqrt{2-2p}},$$

which is $\sqrt{2} - 1 > 0$ at $p = 0$. Thus, the expression attains a maximum at $p \in (0,1)$, meaning the optimal p is greater than 0.

In order for this argument to be valid, we must check one more thing: here, we assumed that it is optimal for the monopolist to sell to both types by assuming that the individual rationality constraint for type 1 customers binds. Suppose we ignore this constraint and decide to sell simply to the "high" type 2 customers. Then, the optimal policy is clearly to charge $p = 0$ and extract all the surplus by charging a fixed cost of $C = \int_0^{\sqrt{2}} 1 - x^2/2 \; dx = 2\sqrt{2}/3$. Then, since there is a mass 1/2 of consumers, the monopolist earns profit $\sqrt{2}/3 < 1/2$. However, consider the case where the monopolist sells to both types, sets $C = 1/2$ and $p = 0$. Then, the monopolist will get a profit of 1/2 from each type of consumer, which will lead to a profit of 1/2 in the aggregate. Thus, by selling to the low type, *even if he must use a two-part tariff*, the monopolist can do no worse than selling to only the high type. Therefore, the analysis above is valid.

In the textbook single-type two-part tariff, the monopolist can always reduce the price and internalize the consumer surplus using the fixed fee. In this case, the fixed fee is bounded above by the willingness to pay of the low types. Increasing the price from marginal cost has a second-order effect for the profits from the low types. The first-order effects are: increasing the price by δ reduces the fixed fee collected (by $\delta D_1(c) = \delta$ from the high types), but it also generates more revenue from each sale from the high types (by $\delta D_2(c) = \delta\sqrt{2}$). Since $\sqrt{2} > 1$, it makes sense to increase the price from marginal cost.

2 Suppose you are the manufacturer of surfboards which are sold in two separate markets: California and Hawaii. You have factories in both locations, and each can produce an unlimited number of surfboards at a constant marginal cost of \$10 per surboard. Over the last fourteen weeks you've conducted an experiment by varying your prices each week. Your sales at various prices were:

a Use an OLS regression to estimate linear demand curves for each market.

a sol The results from the OLS regressions are

$$P^c = 26.251 - 0.1366q^c$$
$$P^h = 35.388 - 0.815q^h$$

b Given these estimated demand curves what prices would you set in each market? How would you change these prices if antitrust laws required that you set a common price across both markets?

b sol If we can set independent prices in each market, then we maximize

$$q(A - bq - 10)$$

for each market. Taking FOC's gives prices around 18 and 23 in California and Hawaii, respectively.

If we were required to set the same prices in both locations, then we are maximizing the following

$$(p - 10)\left(\frac{26.251 - p}{0.1366} + \frac{35.388 - p}{0.815}\right)$$

which occurs at $p = 18.8$.

c How would profits and consumer surplus be affected by the shift to uniform pricing?

c sol Given our linear demand specification, consumer surplus is just the area of the triangle. Hence, CS $= \frac{1}{8b}(A-10)^2$ which yields to CS of 242 and 99 in CA and HI, respectively. Profits are $\frac{1}{8b}(A-10)^2$ which yields to profits of 483 and 198 in CA and HI, respectively.

Under the uniform pricing hypothesis, one can similarly calculate the area of the triangles to get that the profits are 480 178 in CA and HI. Meanwhile CS is 204 169 in CA and HI. Note that aggregate profits are mechanically lower, since uniform pricing just imposes a constraint on the maximization problem. Further, uniform pricing necessarily decreases the price in one market and increases the price in another market. Hence, the CS increases in one market and decreases in another.

d Suppose retailers can ship surfboards between California and Hawaii for \$4 per board. Would this disturb your discriminatory pricing strategy, and if so what would your response be? How is this problem similar to and different from a standard 2nd degree price discrimination model?

d sol Since the price difference between the two markets at the optimum is larger than 4, this would induce an arbitrage opportunity and thus consumers would buy from the retailer rather than you.

Given this arbitrage constraint, it is without loss of optimality to consider a pricing strategy in which the prices in the two markets are no more than four dollars apart. This moves our pricing strategy to be somewhere in between that of uniform pricing and discriminatory.

It is similar to second degree discrimination since you can price based on consumer charachteristics, however now there is an arbitrage constraint.

3 3. Consider a model with consumers uniformly distributed on the interval $[0, 1]$. Two suppliers selling the same good are located at points a and $1 - b$ with $0 \leq a, b \leq \frac{1}{2}$. Their production costs per unit are c_1 and c_2, respectively. Consumers buy zero or one unit of the good. They receive zero utility if they don't buy the good and utility $v - p - tx^2$ if they buy the good from a firm at a distance of x from their location. Assume that the firms choose prices simultaneously, and that their objective is to maximize profits.

a Find the Nash equilibrium prices and profits in this model assuming that v is sufficiently large so that the equilibrium involves all consumers purchasing the good. How large can firm 1's cost disadvantage be if it does make positive profits in equilibrium?

a sol We begin first by finding the location x^* of a consumer indifferent between purchasing from A and B at prices of p_A and p_B, respectively. x^* satisfies

$$v - p_a - t\left(x^* - a\right)^2 = v - p_b - t\left(x^* - (1-b)\right)^2 \Longrightarrow x^* = \frac{p_a - p_b}{2t(a+b-1)} + \frac{1}{2}(a - b + 1)$$

Note that demands for A and B are x^* and $1 - x^*$, respectively. Assume that $x^* \in (0,1)$. Then, profits are given by

$$\begin{aligned} \Pi_A &= (p_A - c_A)\, x^* \\ \Pi_B &= (p_B - c_B) + (1 - x^*) \end{aligned}$$

Taking first order conditions and solving for p_A, p_B yields

$$\begin{aligned} p_A &= \frac{1}{3}\left(t\left(-a^2 - 2a + b^2 - 4b + 3\right) + 2c_a + c_b\right) \\ p_B &= \frac{1}{3}\left(t\left(-b^2 - 2b + a^2 - 4a + 3\right) + 2c_b + c_a\right) \end{aligned}$$

A has positive profits whenever $p_A - c_A > 0$; this occurs precisely when

$$c_A - c_B < t\left(-a^2 - 2a + b^2 - 4b + 3\right)$$

b Suppose that before choosing prices the firms play a first period game where they simultaneously choose where to locate. Assume that the firms costs are equal, $c_1 = c_2 = c$. Show that in equilibrium the firms are maximally differentiated.

b sol We can now calculate the profits for firm a, and there will be a symmetric expression for firm b.

$$\Pi_A = \frac{1}{6}\left(-a^2 - 2a + b^2 - 4b + 3\right)(a - b + 3)$$

One can now take the derivative of this expression w.r.t. to a and see that the unique equilibrium will be maximal differentiation.

4 Consider the model of vertical differentiation discussed in class (and in section 7.5.1 of Tirole). Suppose that the firms' costs are higher than I assumed so that the equilibrium prices end up being such that some consumers do not buy the product. Write down the equations for demand when prices are such that the highest value consumers buy from firm H, some buy from firm L and some do not buy at all. Assuming that the best responses are always given by the first order conditions obtained by maximizing relative to these demands find the best response functions and solve for the Nash equilibrium. For what values of c do the equations you've written really give the Nash equilibrium of the game.

sol Consider the same setup as the vertical differentiation model in class. However, now assume that $\underline{\theta} < \theta' < \widehat{\theta} < \bar{\theta}$. Where θ' is the indifferent consumer between buying and not buying. Demands are then given by

$$D_H = \bar{\theta} - \widehat{\theta} = \bar{\theta} - \frac{p_H - p_L}{s_H - s_L}$$
$$D_L = \widehat{\theta} - \theta' = \frac{p_H - p_L}{s_H - s_L} - \frac{p_L}{s_L}$$

and so best response functions are given by

$$BR_H\left(p_L\right) = \arg\max_p (p - c)\left(\bar{\theta} - \frac{p - p_L}{s_H - s_L}\right)$$
$$BR_L\left(p_H\right) = \arg\max_p (p - c)\left(\frac{p_H - p}{s_H - s_L} - \frac{p}{s_L}\right)$$

Taking the FOCs and solving, we find that

$$p_H = \frac{1}{4s_H - s_L}\left(3cs_H + 2s_H^2\bar{\theta} - 2s_H s_L\bar{\theta}\right)$$
$$p_L = \frac{1}{4s_H - s_L}\left(2cs_H + cs_L + s_H s_L\bar{\theta} - s_L^2\bar{\theta}\right)$$

The conditions for this to be a valid solution are: - $p_H, p_L > c$ - $\underline{\theta} < \theta' < \hat{\theta} < \bar{\theta}$

5 Open ended.