14.30 PROBLEM SET 7 SUGGESTED ANSWERS

Problem 1

a.   For the statistic to be unbiased we must have

$$\mu = E\left(\sum_{i=1}^{n} c_i X_i\right)$$

$$= E\left(X_i\right)\sum_{i=1}^{n} c_i$$

$$= \mu\sum_{i=1}^{n} c_i$$

So the statistic will be unbiased if and only if $\sum_{i=1}^{n} c_i = 1$.

b.   Given that all of the X_i are independent, the variance of the statistic is

$$Var\left(\sum_{i=1}^{n} c_i X_i\right) = \sum_{i=1}^{n} c_i^2 Var\left(X_i\right)$$

$$= \sigma^2 \sum_{i=1}^{n} c_i^2$$

c.   We want to minimize

$$\sigma^2 \sum_{i=1}^{n} c_i^2$$

subject to the constraint

$$\sum_{i=1}^{n} c_i = 1$$

The Lagrangian of this optimization problem is

$$L = \sigma^2 \sum_{i=1}^{n} c_i^2 - \lambda\left(\sum_{i=1}^{n} c_i - 1\right)$$

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And we have the first-order conditions

$$\frac{\partial L}{\partial c_i} = \sigma^2 2 c_i - \lambda = 0$$

$$c_i = \frac{\lambda}{2\sigma^2}$$

This implies that all of the c_i are the same, that is $c_1 = c_2 = ... = c_n = c$. We can then solve for this common value c by substituting back into the constraint $\sum_{i=1}^{n} c = 1$, which implies that $c = \frac{1}{n}$.

Problem 2

a.

$$
\begin{aligned}
E\left(\overline{X}\right) &= E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}E\left(X_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\mu \\
&= \mu
\end{aligned}
$$

So $\overline{X}$ is an unbiased estimator of μ.

b.

$$
\begin{aligned}
MSE\left(\overline{X}\right) &= \left(E\left(\overline{X}\right) - \mu\right)^2 + Var\left(\overline{X}\right) \\
&= 0 + Var\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right) \\
&= \frac{1}{n^2}Var\left(\sum_{i=1}^{n}X_i\right) \\
&= \frac{1}{n^2}\left(\sum_i Var\left(X_i\right) + \sum_i\sum_{j\neq i}Cov\left(X_i, X_j\right)\right) \\
&= \frac{1}{n^2}\left(\sum_i\sigma^2 + \sum_i\sum_{j\neq i}\rho\sigma^2\right) \\
&= \frac{\sigma^2}{n} + \frac{(n-1)\rho\sigma^2}{n}
\end{aligned}
$$

c.   To determine whether the sample mean is a consistent estimator, we need to see whether the MSE goes to zero as n approaches infinity.

$$
\begin{aligned}
\lim_{n\to\infty} MSE\left(\overline{X}\right) &= \lim_{n\to\infty}\left(\frac{\sigma^2}{n} + \frac{(n-1)\rho\sigma^2}{n}\right) \\
&= \lim_{n\to\infty}\frac{\sigma^2}{n} + \lim_{n\to\infty}\frac{(n-1)\rho\sigma^2}{n} \\
&= 0 + \rho\sigma^2 = \rho\sigma^2
\end{aligned}
$$

Thus (assuming a positive variance) the sample mean is not a consistent estimator of the population mean unless $\rho = 0$.

Problem 3

a.   We know that the first moment of Z is

$$
\begin{aligned}
E\left(Z\right) &= \int_0^\infty zf\left(z\right)dz \\
&= \int_0^\infty z\lambda e^{-\lambda z}dz \\
&= \frac{1}{\lambda}
\end{aligned}
$$

Then, we calculate our method of moments estimator by setting $E\left(Z\right) = \overline{Z}$ and solving for λ.

$$
\begin{aligned}
\frac{1}{\lambda} &= \frac{1}{n}\sum_{i=1}^n Z_i \\
\widehat{\lambda}_{MM} &= \frac{n}{\sum_{i=1}^n z_i} = \frac{1}{\overline{Z}}
\end{aligned}
$$

b.   The exponential pdf is $f\left(z\right) = \lambda e^{-\lambda z}$. Thus, the likelihood function for λ is

$$
L(\lambda; z) = \prod_{i=1}^n \lambda e^{-\lambda z_i}.
$$

Then, the log likelihood function is

$$
\begin{aligned}
\ln L(\lambda; z) &= \ln\left(\prod_{i=1}^n \lambda e^{-\lambda z_i}\right) \\
&= \sum_{i=1}^n (-\lambda z_i) + n\ln\lambda
\end{aligned}
$$

Differentiating the function with respect to λ and set it to 0,

$$
\frac{\partial \ln L(\lambda; z)}{\partial \lambda} = -\sum_{i=1}^n (z_i) + \frac{n}{\lambda} = 0
$$

$$
\Longrightarrow
$$

$$
\sum_{i=1}^n z_i = \frac{n}{\lambda}
$$

So we find that the MLE estimator is the same as the MM estimator:

$$\widehat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} z_i} = \frac{1}{\overline{Z}}$$

c.   Using the invariance property, the MLE for $\sqrt{\lambda}$ is simply

$$\widehat{\sqrt{\lambda}}_{MLE} = \sqrt{\widehat{\lambda}_{MLE}} = \frac{1}{\sqrt{\overline{Z}}} = \sqrt{\frac{n}{\sum_{i=1}^{n} Z_i}}.$$

d.   We know that $\widehat{\lambda}_{MLE}$ will be consistent because all MLE estimators are consistent. But is it unbiased? A direct calculation of $E(\widehat{\lambda}_{MLE}) = E\left(\frac{1}{\overline{Z}}\right)$ is untractable (note that $E\left(\frac{1}{\overline{Z}}\right) \neq \frac{1}{E(\overline{Z})}$, in general). But we can use Jensens's inequality to help us out, which tells us that for any random variable X, if $g(x)$ is a convex function, then

$$E(g(X)) \geq g(E(X))$$

with a strict inequality if $g(x)$ is strictly convex. Thus, using $\overline{Z}$ as our random variable and the strictly convex function $g(\overline{Z}) = \frac{1}{\overline{Z}}$, we have

$$E\left(\frac{1}{\overline{Z}}\right) > \frac{1}{E(\overline{Z})} = \frac{1}{E(Z)} = \lambda$$

and we know that our estimator is biased.

Note that if we had instead used the version of the exponetial pdf that replaces λ with $\frac{1}{\beta}$, and estimated the parameter β by either method, we would have found $\widehat{\beta} = \overline{Z}$, which is both unbiased (since $E(Z) = \frac{1}{\lambda} = \beta$) and consistent (by the law of large numbers).

Problem 4

a. $Bias\left[\widehat{\mu}_{1,n}\right] = E\left[\widehat{\mu}_{1,n}\right] - \mu = E\left[X_n\right] - \mu = \mu - \mu = 0$ so $\widehat{\mu}_{1,n}$ is unbiased for every n.

$$Bias\left[\widehat{\mu}_{2,n}\right] = E\left[\widehat{\mu}_{2,n}\right] - \mu = E\left[\frac{1}{n+1}\sum_{i=1}^{n} X_i\right] - \mu = \frac{1}{n+1}\sum_{i=1}^{n} E\left[X_i\right] - \mu =$$

$\frac{n}{n+1}\mu - \mu = \frac{-\mu}{n+1} \neq 0$ so $\widehat{\mu}_{2,n}$ is biased for for every n.

b. $\lim_{n\to\infty} P\left(\left|\widehat{\mu}_{1,n} - \mu\right| < \varepsilon\right) = \lim_{n\to\infty} P\left(\left|X_n - \mu\right| < \varepsilon\right) = P\left(\mu - \varepsilon < X_n < \mu + \varepsilon\right) = P\left(X_n \leq \mu + \varepsilon\right) - P\left(X_n \leq \mu - \varepsilon\right)$ (because the R.V. are continuous)$= F\left(\mu + \varepsilon\right) - F\left(\mu - \varepsilon\right) \neq 0$ for some $\varepsilon > 0$. Thus $\widehat{\mu}_{1,n}$ is not consistent.

As for $\widehat{\mu}_{2,n}$: $Var\left[\widehat{\mu}_{2,n}\right] = Var\left[\frac{1}{n+1}\sum_{i=1}^{n} X_i\right] = \left(\frac{1}{n+1}\right)^2 \sum_{i=1}^{n} Var\left[X_i\right] =$

$\left(\frac{1}{n+1}\right)^2 n\sigma^2$ so:

$\lim_{n\to\infty}\left(MSE\left[\widehat{\mu}_{2,n}\right]\right) = \lim_{n\to\infty}\left(Var\left[\widehat{\mu}_{2,n}\right]\right) + \lim_{n\to\infty}\left(Bias\left[\widehat{\mu}_{2,n}\right]\right)^2 =$ 0, proving that $\widehat{\mu}_{2,n}$ is consistent.

c.   An unbiased estimator is not necessarily consistent; a consistent estimator is not necessarily unbiased.

Problem 5
a.   MM: Since we know that $\theta_l = 0$, we only need to use the first moment equation:

$$E(X) = \frac{0 + \theta_h}{2},$$

Then, the MM estimator is obtained by solving

$$\frac{\widehat{\theta}_h}{2} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$$

$$\widehat{\theta}_h = \frac{2}{n}\sum_{i=1}^{n} X_i = 2\overline{X}$$

MLE: The uniform pdf is

$$f(x) = \frac{1}{\theta_h - \theta_l} = \frac{1}{\theta_h}$$

and the likelihood function and log likelihood functions are given (respectively) by

$$L(0, \theta_h; x_1, x_2, ..., x_n) = \left(\frac{1}{\theta_h}\right)^n$$

$$\ln L(0, \theta_h; x) = -n\ln\theta_h$$

In order to maximize the log likelihood function above, we need to minimize θ_h subject to the constraint $x_i \leq \theta_h \ \forall \ x_i$.
Then, it must be that

$$\widehat{\theta}_h = \max(x_i)$$

b.   The first two moments are

$$E(X) = \frac{\theta_l + \theta_h}{2},$$

$$E(X^2) = \frac{\theta_l^2 + \theta_h^2 + \theta_l\theta_h}{3}.$$

Then, the MM estimators are obtained by solving

$$\frac{\widehat{\theta}_l + \widehat{\theta}_h}{2} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X},$$

$$\frac{\widehat{\theta}_l^2 + \widehat{\theta}_h^2 + \widehat{\theta}_l\widehat{\theta}_h}{3} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \overline{X^2}.$$

They are given by

$$\widehat{\theta}_l = \overline{X} - \sqrt{3(\overline{X^2} - \overline{X}^2)},$$
$$\widehat{\theta}_h = \overline{X} + \sqrt{3(\overline{X^2} - \overline{X}^2)}.$$

c.   We begin with the MM estimator for part a:

$$E\left(\widehat{\theta}_h\right) = E\left(2\overline{X}\right) = 2\frac{\widehat{\theta}_h}{2} = \widehat{\theta}_h$$

So this estimator is unbiased. Then we consider the variance.

$$Var\left(\widehat{\theta}_h\right) = Var\left(2\overline{X}\right) = \frac{4}{n^2}\sum_{i=1}^{n}Var\left(X_i\right) = \frac{4}{n}\left(\frac{\theta_1^2 + \theta_2^2 - 2\theta_1\theta_2}{12}\right)$$

Because the bias is zero and the variance approaches zero as n gets large, the MSE also approaches zero, and the estimator is consistent.

For the MLE estimator, we use the fact (shown in problem 6) that, for this estimator, $f(x) = \frac{nx^{n-1}}{\theta^n}$. Then we can see that the MLE estimator is biased:

$$E\left(\widehat{\theta}_h\right) = \int_0^{\theta_h} \frac{nx^n}{\theta_h^n}dx = \frac{n}{n+1}\theta_h$$

However, as n gets large, $\frac{n}{n+1} \longrightarrow 1$, so the bias approaches zero, and we know that in general, MLE estimators are consistent.

For the MM estimators in part b, finding the expected value is not particularly tractable nor particularly interesting, so I will retract this portion of the question.

Problem 6
Note that for a $U[0,\theta]$ distribution, $f(x) = \frac{1}{\theta}$, $F(x) = \frac{x}{\theta}$, $E(X) = \frac{\theta}{2}$, and $Var(X) = \frac{\theta^2}{12}$.
a.   It will be important that the sample draws are independent (and identical). Then,

$$\begin{aligned}
\Pr\left(X_{(n)} \leq x\right) &= \Pr\left(\max(X_i) \leq x\right) \\
&= \prod_{i=1}^{n} \Pr\left(X_i \leq x\right) \\
&= F(x)^n \\
&= \left(\frac{x}{\theta}\right)^n
\end{aligned}$$

b.   In part a, we found the cdf of $X_{(n)}$, so the pdf is just the derivaive of this:

$$
\begin{aligned}
f_{(n)}(x) &= \frac{d}{dx} F_{(n)}(x) \\
&= \frac{d}{dx} \left(\frac{x}{\theta}\right)^n \\
&= \frac{nx^{n-1}}{\theta^n}
\end{aligned}
$$

c.   We want to choose constants k_1, k_2, and k_3 such that our estimators are unbiased. For the first estimator,

$$
\begin{aligned}
E\left(\widehat{k_1} X_2\right) &= \theta \\
\widehat{k_1} E(X_2) &= \theta \\
\widehat{k_1} \frac{\theta}{2} &= \theta \\
\widehat{k_1} &= 2
\end{aligned}
$$

Then, for the second estimator

$$
\begin{aligned}
E\left(\widehat{k_2} \overline{X}\right) &= \theta \\
\widehat{k_2} E\left(\overline{X}\right) &= \theta \\
\widehat{k_2} \frac{\theta}{2} &= \theta \\
\widehat{k_2} &= 2
\end{aligned}
$$

And for the third estimator

$$
\begin{aligned}
E\left(\widehat{k_3} X_{(n)}\right) &= \theta \\
\widehat{k_3} E\left(X_{(n)}\right) &= \theta \\
\widehat{k_3} \int_0^\theta x \frac{nx^{n-1}}{\theta^n} dx &= \theta \\
\widehat{k_3} \frac{n}{\theta^n} \left(\frac{\theta^{n+1}}{n+1}\right) &= \theta \\
\widehat{k_3} &= \frac{n+1}{n}
\end{aligned}
$$

d.   Now we calculate the variances of our estimators, using the constants that we found in part c.

$$Var\left(2X_2\right) \quad = \quad 4Var\left(X_2\right) = \frac{\theta^2}{3}$$

$$Var\left(2\overline{X}\right) \quad = \quad 4Var\left(\overline{X}\right) = \frac{\theta^2}{3n}$$

$$Var\left(\frac{n+1}{n}X_{(n)}\right) \quad = \quad \left(\frac{n+1}{n}\right)^2 Var\left(X_{(n)}\right)$$

To find $Var\left(X_{(n)}\right)$, we will need $E_{(n)}\left(X^2\right)$:

$$E_{(n)}\left(X^2\right) = \int_0^\theta \frac{nx^{n+1}}{\theta^n}dx = \frac{n}{n+2}\theta^2$$

so

$$Var\left(X_{(n)}\right) = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\right)^2\theta^2 = \frac{n\theta^2}{(n+2)(n+1)^2}$$

and

$$Var\left(\frac{n+1}{n}X_{(n)}\right) = \left(\frac{n+1}{n}\right)^2\frac{n\theta^2}{(n+2)(n+1)^2} = \frac{\theta^2}{n(n+2)}$$

Thus, whenever $n > 1$,

$$Var\left(\widehat{k_3}X_{(n)}\right) < Var\left(\widehat{k_2}\overline{X}\right) < Var\left(\widehat{k_1}X_2\right)$$