14.30 PROBLEM SET 3 - SUGGESTED ANSWERS

Problem 1

a.

$$\begin{aligned} E\left(ag_1\left(X\right)+bg_2\left(X\right)+c\right) &= \int_{-\infty}^{\infty}\left(ag_1\left(x\right)+bg_2\left(x\right)+c\right)f_X\left(x\right)dx \\ &= \int_{-\infty}^{\infty}ag_1\left(x\right)f_X\left(x\right)dx+\int_{-\infty}^{\infty}bg_2\left(x\right)f_X\left(x\right)dx+\int_{-\infty}^{\infty}cf_X\left(x\right)dx \\ &= a\int_{-\infty}^{\infty}g_1\left(x\right)f_X\left(x\right)dx+b\int_{-\infty}^{\infty}g_2\left(x\right)f_X\left(x\right)dx+c\int_{-\infty}^{\infty}f_X\left(x\right)dx \\ &= aE\left[g_1\left(X\right)\right]+bE\left[g_2\left(X\right)\right]+c \end{aligned}$$

b. No, because $g_1\left(x\right)$ is not necessarily linear. For example take $g_1\left(x\right)=x^2, X\sim U\left[0,1\right]$. Here: $E\left(g_1\left(X\right)\right)=\int_0^1 x^2dx=\frac{1}{3}$, but: $g_1\left(E\left[X\right]\right)=\left(\int_0^1 xdx\right)^2=\frac{1}{4}$. Thus: $E\left(g_1\left(X\right)\right)>g_1\left(E\left[X\right]\right)$. More generally, Jensen's Inequality states that for a strictly convex function $g_1\left(E\left[X\right]\right)$ (such as $g_1\left(x\right)=x^2$): $E\left(g_1\left(X\right)\right)>g_1\left(E\left[X\right]\right)$. For a strictly concave function $g_1\left(E\left[X\right]\right): E\left(g_1\left(X\right)\right)<g_1\left(E\left[X\right]\right).$

c.

$$\begin{aligned} Var\left(g_1\left(X\right)+g_2\left(X\right)\right) &= E\left(\left(g_1\left(X\right)+g_2\left(X\right)\right)^2\right)-\left(E\left(g_1\left(X\right)\right)+E\left(g_2\left(X\right)\right)\right)^2 \\ &= E\left(g_1\left(X\right)^2\right)-\left(E\left(g_1\left(X\right)\right)\right)^2 \\ &\quad +E\left(g_2\left(X\right)^2\right)-\left(E\left(g_2\left(X\right)\right)\right)^2 \\ &\quad +2\left(E\left(g_1\left(X\right)g_2\left(X\right)\right)-E\left(g_1\left(X\right)\right)E\left(g_2\left(X\right)\right)\right) \\ &= Var\left(g_1\left(X\right)\right)+Var\left(g_2\left(X\right)\right)+2Cov\left(g_1\left(X\right),g_2\left(X\right)\right) \end{aligned}$$

which aligns with the general formula for the variance of a sum of random variables.

Problem 2

a. $E\left[\left(X_1-2X_2+X_3\right)\right]=E\left[X_1\right]-2E\left[X_2\right]+E\left[X_3\right]=2\left(0.5-2\left(1\right)+1.5\right)=0$

b. We first find $f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \begin{cases} \frac{1}{6} \text{ if } x_1 \in [0,1], x_2 \in [0,2], x_3 \in [0,3] \\ 0 \text{ otherwise} \end{cases}$.
We can then define $g(X_1,X_2,X_3) = (X_1 - 2X_2 + X_3)^2$. It follows that

$$\begin{aligned} E\left[(X_1 - 2X_2 + X_3)^2\right] &= E\left[g(X_1,X_2,X_3)\right] \\ &= \int_0^3 \int_0^2 \int_0^1 \frac{1}{6}(x_1 - 2x_2 + x_3)^2 \, dx_1 dx_2 dx_3 \\ &= \frac{1}{6}\int_0^3 \int_0^2 \left[\frac{1}{3} + 4x_2^2 + x_3^2 - 2x_2 - 4x_2x_3 + x_3\right] dx_2 dx_3 \\ &= \frac{1}{6}\int_0^3 \left[\frac{22}{3} + 2x_3^2 - 6x_3\right] dx_3 \\ &= \frac{1}{6}(22 + 18 - 27) \\ &= \frac{13}{6} \end{aligned}$$

c. $Var[Z] = E\left[Z^2\right] - (E[Z])^2$, so: $Var[(X_1 - 2X_2 + X_3)] = E\left[(X_1 - 2X_2 + X_3)^2\right] - (E[(X_1 - 2X_2 + X_3)])^2 = \frac{13}{6} - 0^2 = \frac{13}{6}$

Problem 3

a. $E[X] = 0, E[Y] = \frac{5}{4} \Rightarrow E[X]E[Y] = 0$. Now: $E[XY] = \frac{1}{2}(-1) + \frac{1}{4}(2) = 0 \Rightarrow Cov(X,Y) = E[XY] - E[X]E[Y] = 0 - 0 = 0 \Rightarrow Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt[2]{Var(X)Var(Y)}} = 0$ so X and Y are uncorrelated.

b. X and Y are not independent, for example: $f_{X,Y}(0,1) = 0 \neq \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = f_X(0) f_Y(1)$.

c. You saw in class that independence implies uncorrelatedness. This exercise proves that two uncorrelated random variables are not necessarily independent.

Problem 4

a. $E[X_1 - Y_1] = E[\ X_1] - E[Y_1] = \left(\frac{1}{4}\right)8 - \frac{1}{2}\left(1 + \frac{3}{2}\right) = \frac{3}{4}$.

b. $\Pr[X_1 - Y_1 > 0] = \Pr[X_1 = 8] = \frac{1}{4}$.

c. $Var(X_1 - Y_1) = Var(X_1) + Var(Y_1) - 2Cov(X_1,Y_1) = 12 + \frac{1}{16} - 0 = \frac{193}{16}$.

d. The expected value of the profit is unaffected (because profit is a linear function of revenue and cost). However, the variance will change because the covariance term in the above expression is no longer equal to zero. To find the covariance, we will first need to compute the joint distribution of X_1 and Y_1. We calculate the joint distribution by multiplying the conditional pdf of $X_1|Y_1$ by the marginal distribution of Y_1. So we have

$$f_{X_1Y_1}(x_1, y_1) = \begin{cases} \frac{5}{16} \text{ for } \left(0, \frac{3}{2}\right) \\ \frac{3}{16} \text{ for } \left(8, \frac{3}{2}\right) \\ \frac{7}{16} \text{ for } (0, 1) \\ \frac{1}{16} \text{ for } (8, 1) \\ 0 \text{ otherwise} \end{cases}$$

Note that we used the law of total probability to calculate $f_{X_1|Y_1=1}(x_1|Y_1 = 1) = \begin{cases} \frac{7}{8} \text{ if } x_1 = 0 \\ \frac{1}{8} \text{ if } x_1 = 8 \\ 0 \text{ otherwise} \end{cases}$. Then we calculate the covariance:

$$\begin{aligned} Cov(X_1, Y_1) &= E(X_1Y_1) - E(X_1)E(Y_1) \\ &= \left(\frac{5}{16}\right)0 + \left(\frac{3}{16}\right)12 + \left(\frac{7}{16}\right)0 + \left(\frac{1}{16}\right)8 - 2\left(\frac{5}{4}\right) \\ &= \frac{1}{4} \end{aligned}$$

Thus the variance of the profit is now equal to $\frac{185}{16}$.

e. The expected value of your share of the combined profit is $E\left[\frac{1}{2}(X_1 - Y_1 + X_2 - Y_2)\right] = \frac{3}{4}$. The probability of making a positive profit is now $\Pr\left[\frac{1}{2}(X_1 - Y_1 + X_2 - Y_2) > 0\right] = \Pr[X_1 = 8 \cup X_2 = 8] = \Pr[X_1 = 8] + \Pr[X_2 = 8] - \Pr[X_1 = X_2 = 8] = \frac{1}{4} + \frac{1}{4} - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{7}{16}$. And because all of the variables are independent, the variance of your share of the profit is $Var\left(\frac{1}{2}(X_1 - Y_1 + X_2 - Y_2)\right) = \frac{1}{4}Var(X_1) + \frac{1}{4}Var(Y_1) + \frac{1}{4}Var(X_2) + \frac{1}{4}Var(Y_2) = \frac{193}{32}$.

Problem 5

There are three variables in this problem:

$$\begin{aligned} N &= \text{number of classes a student is enrolled in} \\ K &= \text{number of assignments due} \\ T &= \text{total time to complete } K \text{ assignments} \end{aligned}$$

We know that K has a binomial distribution for a given value of N, with $p = 0.7$ and that $T|K$ has the pdf $\frac{1}{2K}e^{\frac{-t}{2K}}$ for $t \geq 0$.

a. We are given that $N = 4$, and want to find the probability that $K = 2$. We use the binomial distribution

$$\Pr(K = 2|N = 4) = \binom{4}{2}(0.7)^2(0.3)^2 = 0.2646$$

b. Now we want the expected value of T, given that $K = 2$.

$$\begin{aligned}
E(T|K=2) &= \int_0^\infty \frac{t}{2K} e^{\frac{-t}{2K}}\, dt \\
&= \int_0^\infty \frac{t}{4} e^{\frac{-t}{4}}\, dt \\
&= \left[-te^{\frac{-t}{4}}\right]_0^\infty + \int_0^\infty e^{\frac{-t}{4}}\, dt \\
&= \left[-te^{\frac{-t}{4}} - 4e^{\frac{-t}{4}}\right]_0^\infty \\
&= 4
\end{aligned}$$

c. It will be most straight-forward to calculate the expected value of T by using the law of iterated expectations.

$$E(T) = E_K(E_T(T|K))$$

We can generalize from part b to see that $E(T|K) = 2K$. We use the fact that N is fixed at 4 and get

$$\begin{aligned}
E(T) &= E_K(2K) \\
&= \sum_{K=0}^{4} 2K \binom{4}{K} (0.7)^K (0.3)^{4-K} \\
&= 5.6
\end{aligned}$$

We will use the conditional variance identity to find $Var(T)$

$$Var(T) = E(Var(T|K)) + Var(E(T|K))$$

But first, we need to find $Var(T|K)$. First we need $E(T^2|K)$.

$$\begin{aligned}
E\left(T^2|K\right) &= \int_0^\infty \frac{t^2}{2K} e^{\frac{-t}{2K}}\, dt \\
&= \left[-t^2 e^{\frac{-t}{2K}}\right]_0^\infty + \int_0^\infty 2te^{\frac{-t}{2K}}\, dt \\
&= \left[-t^2 e^{\frac{-t}{2K}}\right]_0^\infty + \left[-4Kte^{\frac{-t}{2K}}\right]_0^\infty + 4K \int_0^\infty e^{\frac{-t}{2K}}\, dt \\
&= \left[-t^2 e^{\frac{-t}{2K}}\right]_0^\infty + \left[-4Kte^{\frac{-t}{2K}}\right]_0^\infty + \left[-8K^2 e^{\frac{-t}{2K}}\right]_0^\infty \\
&= 8K^2
\end{aligned}$$

So $Var(T|K) = E(T^2|K) - (E(T|K))^2 = 4K^2$
And now

$$\begin{aligned}
Var(T) &= E(4K^2) + Var(2K) \\
&= 8E\left(K^2\right) - 4(E(K))^2 \\
&= 8(8.68) - 4(2.8)^2 \\
&= 38.08
\end{aligned}$$

Problem 6

In parts b-d, plans (with a lower-case p) should be replaced with stocks. Sorry about the typo.

a. Plan 1: $E(10X) = 10E(X) = 10\mu$

Plan 2: $E(5X + 5Y) = 5E(X) + 5E(Y) = 10\mu$

Thus, both plans yield the same expected return.

b. $Var(5X + 5Y) = 25Var(X) + 25Var(Y) - 50Cov(X, Y) = 50\sigma^2$

c. First note that $\rho = \frac{Cov(X,Y)}{\sqrt{\sigma^2}\sqrt{\sigma^2}}$ implies $Cov(X, Y) = \frac{\sigma^2}{2}$ for $\rho = \frac{1}{2}$. Thus, $Var(5X + 5Y) = 25Var(X) + 25Var(Y) - 50Cov(X, Y) = 50\sigma^2 + 50\frac{\sigma^2}{2} = 75\sigma^2$.

d. Since both plans have the same expected return, a risk-averse investor will prefer the one with the lower variance of the return. For Plan 1, we have $Var(10X) = 100\sigma^2$. For Plan 2, we have $Var(5X + 5Y) = 25Var(X) + 25Var(Y) + 50Cov(X, Y) = 50\sigma^2 + 50\rho\mu^2$. Because $-1 \leq \rho \leq 1$, the variance of the return to Plan 2 is weakly less than the variance of the return to Plan 1. Thus even for an unknown ρ, Plan 2 is preferred.