Lecture Note 6 *

Special Distributions (Discrete and Continuous)

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15 Discrete Distributions

We have already seen the binomial distribution and the uniform distribution.

15.1 Hypergeometric Distribution

Let the RV X be the total number of "successes" in a sample of n elements drawn from a population of N elements with a total number of M "successes." Then, the pmf of X, called hypergeometric distribution, is given by:

$$f(x) = P(X = x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \quad \text{for } x = 0, 1, ..., n. \tag{40}$$

With mean and variance:

$$E(X) = \frac{nM}{N} \qquad \text{and} \qquad \text{Var}(X) = \left(\frac{N-n}{N-1}\right)\frac{nM}{N}\left(1 - \frac{M}{N}\right)$$

*Caution: These notes are not necessarily self-explanatory notes. They are to be used as a complement to (and not as a substitute for) the lectures.

15.2 Negative Binomial Distribution

The binomial distribution counts the number of successes in a fixed number of trials (n). Suppose that, instead, we count the number of trials required to get a fixed number of successes (r).

Let the RV X be the total number of trials required to get r "successes." The pmf of X, called negative binomial distribution, is given by:

$$f(x) = P(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \qquad \text{for } x = r, r+1, r+2, ... \tag{41}$$

With mean and variance:

$$E(X) = \frac{r}{p} \qquad \text{and} \qquad \text{Var}(X) = \frac{r(1-p)}{p^2}$$

- $r = 1 \rightarrow$ Geometric distribution: "waiting for the success."

15.3 Poisson Distribution

A RV X is said to have a Poisson distribution with parameter λ ($\lambda > 0$) if the pmf of X is:

$$X \sim \mathcal{P}(\lambda): \qquad f(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{for } x = 0, 1, 2, ... \tag{42}$$

With mean and variance:

$$E(X) = \lambda \qquad \text{and} \qquad \text{Var}(X) = \lambda$$

- λ can be interpreted as a rate per unit of time or per unit of area.

- If X_1 and X_2 are independent RVs that have a Poisson distribution with means λ_1 and λ_2, respectively, then the RV $Y = X_1 + X_2$ has a Poisson distribution with mean $\lambda_1 + \lambda_2$ (function of RVs, Lecture Note 5).

- Note: $\sum_{x=0}^{\infty} f(x) = e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!}}_{=e^{\lambda}} = 1.$

- The Poisson distribution is not derived from a natural experiment, as with the two previous distributions.

Example 15.1. Let X be distributed Poisson (λ). Compute the $E(X)$.

Example 15.2. Assume the number of customers that visit a store daily is a random variable distributed Poisson(λ). It is known that the store receives on average 20 customers per day, so $\lambda = 20$. What is the probability i) that tomorrow there will be 20 visits? ii) that during the next 2 days there will be 30 visits? iii) that tomorrow before midday there will be at least 7 visits?

15.3.1 Poisson Distribution and Poisson Process

A common source of confusion...

A Poisson process with rate λ per unit time is a counting process that satisfies the following two properties:

i) The number of arrivals in every fixed interval of time of length t has a Poisson distribution for which the mean is λt.

ii) The numbers of arrivals in every two disjoint time intervals are independent.

- Poisson process: Use λt when your experiment covers t units.

Example 15.3. Answer Example 15.2 assuming now that the number of customers that visit a certain store follows a Poisson process (with the same average of 20 visits per day).

- **Poisson v/s binomial approach**. As $n \to \infty$, $p \to 0$, and $np \to \lambda$, the limit of the binomial distribution $\longrightarrow$ Poisson distribution.

16 Continuous Distributions

We have already seen the the uniform distribution.

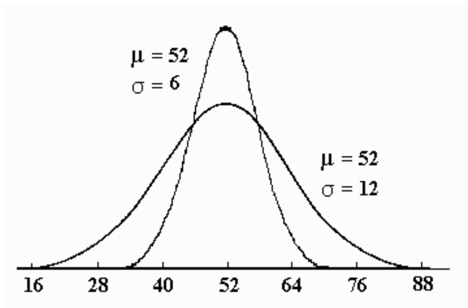
16.1 Normal Distribution

A RV X is said to have a Normal distribution with parameters μ and σ^2 ($\sigma^2 > 0$), if the pdf of X is:

$$X \sim N(\mu, \sigma^2): \qquad f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, \qquad \text{for } -\infty < x < \infty \tag{43}$$

With mean, variance, and MGF:

$$E(X) = \mu, \qquad \text{Var}(X) = \sigma^2, \qquad \text{and} \qquad E(e^{tX}) = e^{t\mu + \frac{\sigma^2 t^2}{2}}$$



- Why is the Normal distribution so important?

  1. The Normal distribution has a familiar bell shape. It gives a theoretical base to the empirical observation that many random phenomena obey, at least approximately, a normal probability distribution:

     > "The further away any particular outcome is from the mean, it is less likely to occur; this characteristic is symmetric whether the deviation is above or below the mean."

     Examples: height or weight of individuals in a population; error made in measuring a physical quantity; level of protein in a particular seed; etc.

2. The Normal distribution gives a good approximation to other distributions, such as the Poisson and the Binomial.

3. The Normal distribution is analytically much more tractable that other bell shape distribution.

4. Central limit theorem (more on this later in LN7).

5. The Normal distribution is very helpful to represent population distributions (linked to point 1).

- Graphic properties.

  1. Bell shape and symmetric.
  2. Centered in the mean (μ), which coincides with the median.
  3. Dispersion/flatness only depends on the variance (σ^2).
  4. $P(\mu - \sigma < X < \mu + \sigma) = 0.6826 \qquad \forall\ \mu, \sigma^2!$
  5. $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544 \qquad \forall\ \mu, \sigma^2!$

- If $X \sim N(\mu, \sigma^2)$, then the RV $Z = (X - \mu)/\sigma$ is distributed $Z \sim N(0, 1)$. This distribution, $N(0, 1)$, is called standard normal distribution, and sometimes its cdf is denoted $F_Z(z) = \Phi(z)$.

- The cdf of the normal distribution does not have an anaclitic solution and its values must be looked up in a $N(0, 1)$ table (see attached table).

- Note that $\Phi(z) = 1 - \Phi(-z)$. In fact: $F_Y(y) = 1 - F_Y(-y) \quad \forall Y \sim N(0, \sigma^2)$.

- If $X_i \sim N(\mu_i, \sigma_i^2)$ and all n X_i are mutually independent, then the RV H is distributed:

$$H = \sum_{i=0}^{n} a_i X_i + b_i \sim N(\sum_{i=0}^{n} a_i \mu_i + b_i \ , \ \sum_{i=0}^{n} a_i^2 \sigma_i^2). \tag{44}$$

Example 16.1. Using the tools developed in Lecture Note 5, derive the distribution of $Z = (X - \mu)/\sigma$ as a transformation of the RV $X \sim N(\mu, \sigma^2)$.

Example 16.2. Compute $E(X)$ where $X \sim N(\mu, \sigma^2)$.

Example 16.3. Assume that the RV X has a normal distribution with mean 5 and standard deviation 2. Find $P(1 < X < 8)$ and $P(|X - 5| < 2)$.

Example 16.4. Assume two types of light bulbs (A and B). The life of bulb type A is distributed normal with mean 100 (hours) and variance 16. The life of bulb type B is distributed normal with mean 110 (hours) and variance 30. i) What is the probability that bulb type A lasts for more than 110 hours? ii) If a bulb type A and a bulb type B are turned on at the same time, what is the probability that type A lasts longer than type B? iii) What is the probability that both bulbs last more than 105 hours?

- The binomial distribution can be approximated with a normal distribution. Rule of thumb: $\min(np, n(1-p)) \geq 5$.

16.2 LogNormal Distribution

If X is a RV and the $ln(X)$ is distributed $N(\mu, \sigma^2)$, then X has a lognormal distribution with pdf (RV transformation):

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{x}e^{-(ln(x)-\mu)^2/(2\sigma^2)}, \qquad \text{for } 0 < x < \infty, \quad -\infty < \mu < \infty,\ \sigma > 0 \tag{45}$$

$$ln(X) \sim N(\mu, \sigma^2) \longleftrightarrow X \sim LnN(\mu, \sigma^2)$$

With mean and variance:

$$E(X) = e^{\mu+(\sigma^2/2)} \qquad \text{and} \qquad \text{Var}(X) = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}.$$

- If $X \sim N(\mu, \sigma^2)$, then $e^X \sim LnN(\mu, \sigma^2)$.

16.3 Gamma Distribution

A RV X is said to have a gamma distribution with parameters α and β $(\alpha, \beta > 0)$ if the pdf of X is:

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}, \qquad \text{for } 0 < x < \infty \tag{46}$$

where,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx \quad \text{finite if } \alpha > 0.$$

$$\Gamma(\alpha) = (\alpha-1)! \text{ if } \alpha \text{ is a positive integer}, \quad \text{and} \quad \Gamma(0.5) = \pi.$$

With mean and variance:

$$E(X) = \alpha\beta \qquad \text{and} \qquad \text{Var}(X) = \alpha\beta^2$$

- Assume a Poisson process. Let Y have a Poisson distribution with parameter λ. Denote X as the waiting time for the r^{th} event to occur. Then, X is distributed gamma with parameters $\alpha = r$ and $\beta = 1/\lambda$.

16.4 Exponential Distribution

A RV X is said to have an exponential distribution with parameter β ($\beta > 0$) if the pdf of X is:

$$f(x) = \frac{1}{\beta}e^{-x/\beta}, \qquad \text{for } 0 < x < \infty \tag{47}$$

With mean and variance:

$$E(X) = \beta \qquad \text{and} \qquad \text{Var}(X) = \beta^2$$

- The exponential distribution is a gamma distribution with $\alpha = 1$.

16.5 Chi-squared Distribution

A RV X is said to have an chi-squared distribution with parameter $p > 0$ (degrees of freedom) if the pdf of X is:

$$X \sim \chi^2_{(p)}: \qquad f(x) = \frac{1}{\Gamma(p/2)2^{p/2}}x^{p/2-1}e^{-x/2}, \qquad \text{for } 0 < x < \infty \text{ and } p \text{ integer.} \tag{48}$$

With mean and variance:

$$E(X) = p \qquad \text{and} \qquad \text{Var}(X) = 2p$$

- The chi-squared distribution is a gamma distribution with $\alpha = p/2$ and $\beta = 2$.

- If $Y \sim N(0,1)$, then the RV $Z = Y^2$ is distributed:

$$Z = Y^2 \sim \chi^2_{(1)} \qquad \text{(random variable transformation.)} \tag{49}$$

- If $X_1 \sim \chi^2_{(p)}$ and $X_2 \sim \chi^2_{(q)}$ are independent, then the RV $H = X_1 + X_2$ is distributed:

$$H = X_1 + X_2 \sim \chi^2_{(p+q)} \qquad \text{(random vector transformation).} \tag{50}$$

- Extensively used in Econometrics.

- Concept of single distribution vs. family of distributions (indexed by one or more parameter).

16.6 Bivariate Normal Distribution

A bivariate random vector (X_1, X_2) is said to have a bivariate normal distribution if the pdf of (X_1, X_2) is:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-b/(2(1-\rho^2))} \tag{51}$$

$$\rho = \text{Corr}(X_1, X_2)$$

$$b \equiv \frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}$$

- $\rho = 0 \longleftrightarrow X_1$ and X_2 independent (only in the normal case) $\longleftrightarrow f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$.