14.30 PROBLEM SET 6 SUGGESTED ANSWERS

Problem 1

a. Let $\widetilde{X}$ denote the sum of the weight of the 100 sampled coins: $\widetilde{X} = \sum_{i=1}^{100} X_i$. Now, $\widetilde{X}$ must be distributed normally, because it is a linear combination of independent normal random variables. Then $E\left(\widetilde{X}\right) = E\left(\sum_{i=1}^{100} X_i\right) = \sum_{i=1}^{100} E(X_i) = 12800$, and $Var\left(\widetilde{X}\right) = Var\left(\sum_{i=1}^{100} X_i\right) = \sum_{i=1}^{100} Var(X_i) = 100$, since the coins are assumed to be independent. Thus, if the Master is honest, $\widetilde{X} \sim N(12800, 100)$.

We must choose some remedy R, such that we punish the Master whenever $\widetilde{X} \leq 12800 - R$, and we want the probability that an honest Master is punished be no greater than 0.01. So we want to find R that satisfies

$$\begin{aligned}
\Pr\left(\widetilde{X} \leq 12800 - R\right) &= 0.01 \\
\Pr\left(\frac{\widetilde{X} - 12800}{10} \leq -\frac{R}{10}\right) &= 0.01 \\
\Pr\left(Z \leq -\frac{R}{10}\right) &= 0.01
\end{aligned}$$

We can use our table to find that $-\frac{R}{10} = -2.33$, which implies $R = 23$.

b. Under the dishonest master, $\widetilde{X} \sim N(12790, 100)$. For $R = 23$, the probability of catching this master is $\Pr\left(\widetilde{X} \leq 12777\right) = \Pr\left(\frac{\widetilde{X}-12800}{10} \leq -\frac{12777-12800}{10}\right) = \Pr(Z \leq -1.3) \approx 0.097$. Thus, the chance that this dishonest Master eludes punishment is greater than 90%.

Problem 2

a. $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$, so $E\left(S^2\right) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right)$

$$
\begin{aligned}
E\left(S^2\right) &= E\left(\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right) \\
&= \frac{1}{n-1}E\left(\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right) \\
&= \frac{1}{n-1}E\left(\sum_{i=1}^{n}\left(X_i^2 - 2X_i\overline{X} + \overline{X}^2\right)\right) \\
&= \frac{1}{n-1}E\left(\sum_{i=1}^{n}X_i^2 - 2\overline{X}\sum_{i=1}^{n}X_i + n\overline{X}^2\right) \\
&= \frac{1}{n-1}E\left(\sum_{i=1}^{n}X_i^2 - 2n\overline{X}^2 + n\overline{X}^2\right) \\
&= \frac{1}{n-1}E\left(\sum_{i=1}^{n}X_i^2 - n\overline{X}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}E\left(X_i^2\right) - nE\left(\overline{X}^2\right)\right) \\
&= \frac{1}{n-1}\left(n\sigma^2 + n\mu^2 - n\left(Var\left(\overline{X}\right) + \mu^2\right)\right) \\
&= \frac{1}{n-1}\left(n\sigma^2 - n\frac{\sigma^2}{n}\right) \\
&= \frac{n-1}{n-1}\left(\sigma^2\right) \\
&= \sigma^2
\end{aligned}
$$

b. To show $\overline{X} \xrightarrow{p} \mu$, we want to prove that for any $\varepsilon > 0$, as $n \to \infty$, $\Pr\left(\left|\overline{X} - \mu\right| < \varepsilon\right) \to 1$. We will denote the mean of a random sample of size n by $\overline{X}_n$.

$$\begin{aligned}
\Pr\left(\left|\overline{X}_n - \mu\right| < \varepsilon\right) &= \Pr\left(-\varepsilon < \overline{X}_n - \mu < \varepsilon\right) \\
&= 1 - 2\Pr\left(\overline{X}_n - \mu > \varepsilon\right) \\
&= 1 - 2\Pr\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} > \frac{\varepsilon}{\sigma/\sqrt{n}}\right) \\
&= 1 - 2\Pr\left(Z > \sqrt{n}\frac{\varepsilon}{\sigma}\right) \\
&= 1 - 2\left(1 - \Pr\left(Z < \sqrt{n}\frac{\varepsilon}{\sigma}\right)\right) \\
&= -1 + 2\Pr\left(Z < \sqrt{n}\frac{\varepsilon}{\sigma}\right)
\end{aligned}$$

Now, as $n \to \infty$, for any $\varepsilon > 1$, $\Phi\left(\sqrt{n}\frac{\varepsilon}{\sigma}\right) \to 1$, so $\Pr\left(\left|\overline{X}_n - \mu\right| < \varepsilon\right) \to -1 + 2 = 1$.

Problem 3

a. Expand the expression for the price of the stock:

$$S_t = S_0 + \sum_{i=1}^{t} X_i$$

Thus the change in the stock's price after 700 periods is

$$\Delta S = S_{700} - S_0 = \sum_{i=1}^{700} X_i$$

b. Define

$$\overline{X} = \frac{1}{700}\sum_{i=1}^{700} X_i$$

Since 700 is "large," the Central Limit Theorem implies that

$$\begin{aligned}
\frac{\sqrt{700}\left(\overline{X} - \mu\right)}{\sigma} &\sim N(0,1) \\
\mu &= 0.39(1) + 0.20(0) + 0.41(-1) \\
&= -0.02 \\
E\left(X_i^2\right) &= 0.39(1)^2 + 0.20(0)^2 + 0.41(-1)^2 \\
&= 0.80 \\
\sigma^2 &= 0.80 - (-0.02)^2 \\
&= 0.7996
\end{aligned}$$

Therefore

$$\overline{X} \sim N\left(-0.02, \frac{0.7996}{700}\right)$$

c. The probability that the stock is up at least 10 over the first 700 periods can be calculated as follows:

$$\begin{aligned}
\Pr\left(\sum_{i=1}^{700} X_i \geq 10\right) &= \Pr\left(\overline{X} \geq \frac{1}{70}\right) \\
&= \Pr\left(\frac{\overline{X}+0.02}{\sqrt{0.7996/700}} \geq \frac{\frac{1}{70}+0.02}{\sqrt{0.7996/700}}\right) \\
&= \Pr\left(Z \geq 1.01\right) \\
&= 1-\Pr\left(Z \leq 1.01\right) \\
&\approx 0.156
\end{aligned}$$

Problem 4

We can denote the weights of each of the hundred booklets as $X_1, ...X_{100}$. This is a random sample from a population with mean 1 and standard deviation 0.05. We want to know the probability that the sum of the X_i is greater than 100.4:

$$\Pr\left(\sum_{i=1}^{100} X_i \geq 100.4\right) = \Pr\left(\overline{X} \geq 1.004\right)$$

The sample mean has the properties such that

$$\begin{aligned}
E\left(\overline{X}\right) &= E\left(X_i\right) = 1 \\
St.\ Dev.\left(\overline{X}\right) &= \sqrt{Var\left(\overline{X}\right)} = \sqrt{\frac{Var(X_i)}{100}} = \frac{St.\ Dev\left(X_i\right)}{10} = 0.005
\end{aligned}$$

Then, by the central limit theorem, we know that (approximately)

$$\overline{X} \sim N\left(1, (0.005)^2\right)$$

Therefore,

$$\begin{aligned}
\Pr\left(\overline{X} \geq 1.004\right) &= \Pr\left(Z \geq \frac{1.004-1}{0.005}\right) \\
&= \Pr\left(Z \geq 0.8\right) \\
&\approx 0.2119
\end{aligned}$$

where Z is a standard normal random variable.