# 0 (Four) Most Common Hypothesis Tests: An Applied Review

We review the four most common hypothesis test structures applying them to the same random sample example. We change the hypotheses and the decision rule in each test.

## 0.1 Random Sample

Let $X_1, ..., X_{10}$ be a random sample from a normal population with unknown mean ($\mu$) and known standard deviation ($\sigma = 1$). The following table presents the realization of the random sample.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $\bar{x}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 0.4 | -0.3 | 1.2 | -0.6 | 1.6 | -0.3 | 2.0 | 0.5 | -0.1 | 0.54 |

## 0.2 Likelihood Ratio Test (LRT):

Using a LR Test of size 5%, test the null hypothesis that the population mean is 0 against the alternative hypothesis that the population mean is 1.

$$H_0: \quad \mu = 0$$
$$H_1: \quad \mu = 1$$

Decision Rule form: "Reject $H_0$ if $f_1(\mathbf{x})/f_0(\mathbf{x}) > k$".

• We need to compute $k$ and $f_i(\mathbf{x})$ for $i = 0, 1$. The value of $k$ depends on the size of the test that we want to construct and the value of $f_i(\mathbf{x})$ depends on the realization of the random sample.

#### 0.2.1 Computing $k$

To compute $k$ we need to find the distribution of $f_1(\mathbf{X})/f_0(\mathbf{X})$ under the null hypothesis. Note that the capital letter for $\mathbf{X}$ represents the fact that the statistic $f_1(\mathbf{X})/f_0(\mathbf{X})$ is a random variable constructed as a function of the random variables in the random sample.

The way to compute the distribution of $f_1(\mathbf{X})/f_0(\mathbf{X})$ depends on the population distributions assumed (case specific) and many times it is very hard to compute. Once we know the distribution of $f_1(\mathbf{X})/f_0(\mathbf{X})$, we look for the value of $k$ that satisfy the size-condition:

$$P\Big(f_1(\mathbf{X})/f_0(\mathbf{X}) > k|\mu = 0\Big) = \alpha = 0.05$$

DeGroot and Schervish (2002, pages 465-8) propose a strategy that allows us to compute $k$ without necessarily knowing the distribution of $f_1(\mathbf{X})/f_0(\mathbf{X}$. Using this method, we find that $k = 1.22$, which implies that $P\big(f_1(\mathbf{X})/f_0(\mathbf{X}) > 1.22|\mu = 0\big) = 0.05$. I leave you to check the details in DeGroot and Schervish's book (not necessary for the exam).

#### 0.2.2 Computing the likelihood ratio $f_1(\mathbf{x})/f_0(\mathbf{x})$

- Computing $f_0(\mathbf{x})$:

$$\begin{aligned} f_0(\mathbf{x}|\mu = 0) &= \prod_{j=1}^{n} f_0(x_j|\mu = 0) = \prod_{j=1}^{10} \left( \frac{1}{\sqrt{2\pi}\sigma} \; e^{-(x-\mu)^2/(2\sigma^2)} \mid \mu = 0, \;\sigma = 1 \right) \\ &= \frac{1}{(2\pi)^{10/2} \cdot 1^{10}} \; e^{-(2 \cdot 1^2)^{-1} \sum_{j=1}^{10} (x_j - 0)^2} = \frac{1}{(2\pi)^{10/2} \cdot 1^{10}} \; e^{-(2 \cdot 1^2)^{-1} 9.96} \end{aligned}$$

- Computing $f_1(\mathbf{x})$:

$$\begin{aligned} f_1(\mathbf{x}|\mu = 1) &= \prod_{j=1}^{n} f_0(x_j|\mu = 1) = \prod_{j=1}^{10} \left( \frac{1}{\sqrt{2\pi}\sigma} \; e^{-(x-\mu)^2/(2\sigma^2)} \mid \mu = 1, \;\sigma = 1 \right) \\ &= \frac{1}{(2\pi)^{10/2} \cdot 1^{10}} \; e^{-(2 \cdot 1^2)^{-1} \sum_{j=1}^{10} (x_j - 1)^2} = \frac{1}{(2\pi)^{10/2} \cdot 1^{10}} \; e^{-(2 \cdot 1^2)^{-1} 9.16} \end{aligned}$$

$\Longrightarrow \;\; f_1(\mathbf{x})/f_0(\mathbf{x}) = e^{-0.5 \cdot 9.16}/e^{-0.5 \cdot 9.96} = e^{0.40} = 1.49$

#### 0.2.3 Result of the test

$f_1(\mathbf{x})/f_0(\mathbf{x}) = 1.49 > k = 1.22$, so we can reject at 5% the null hypothesis that the population mean is 0 against the alternative hypothesis that the population mean is 1.

### 0.3 One-sided Test:

Using a One-sided Test of size 6%, test the null hypothesis that the population mean is 0.4 against the alternative hypothesis that the population mean is higher than 0.4.

$$H_0: \quad \mu = 0.4$$
$$H_1: \quad \mu > 0.4$$

Decision Rule form: "Reject $H_0$ if $\bar{x} > c$".

• We need to compute $c$, which depends on the size of the test that we want to construct.

#### 0.3.1 Computing $c$

We need to find the value of $c$ that satisfy the condition that the probability of type 1 error is 6%.

$$P\Big(\bar{X} > c|\mu = 0.4\Big) = \alpha = 0.06$$

To compute $c$ we need to find the distribution of the random variable $\bar{X}$. The random sample is normal, so we know that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$. Under the null hypothesis $\bar{X} \sim N(0.4, \frac{1^2}{10})$, and therefore,

$$P\left(\frac{\sqrt{10}(\bar{X} - 0.4)}{1} > \frac{\sqrt{10}(c - 0.4)}{1}\right) = 0.06$$

Since $\frac{\sqrt{10}(\bar{X}-0.4)}{1} = Z \sim N(0,1)$, we know that

$$P\left(Z > \frac{\sqrt{10}(c - 0.4)}{1}\right) = 0.06$$

Since $P(Z > z_{0.94}) = 0.06$, where $z_{0.94} = 1.555$, we know that

$$\frac{\sqrt{10}(c - 0.4)}{1} = 1.555$$

and that

$$c = 0.4 + \frac{1.555}{\sqrt{10}} = 0.892$$

#### 0.3.2 Result of the test

$\bar{x} = 0.54 < c = 0.892$, so we cannot reject at 6% the null hypothesis that the population mean is 0.4 against the alternative hypothesis that the population mean is higher than 0.4.

### 0.4 Two-sided Test:

Using a symmetric Two-sided Test of size 1%, test the null hypothesis that the population mean is 0.1 against the alternative hypothesis that the population mean is different than 0.1.

$$H_0: \quad \mu = 0.1$$
$$H_1: \quad \mu \neq 0.1$$

Decision Rule form: "Reject $H_0$ if $\bar{x} < c_1$ or if $\bar{x} > c_2$".

- We need to compute $c_1$ and $c_2$, which depend on the size of the test.

#### 0.4.1 Computing $c_1$ and $c_2$

We need to find the value of $c_1$ and $c_2$ that satisfy the condition that the probability of type 1 error is 1%.

$$P\Big(\bar{X} < c_1 | \mu = 0.1\Big) + P\Big(\bar{X} > c_2 | \mu = 0.1\Big) = \alpha = 0.01$$

Since we are constructing a symmetric test, we have to satisfy the following two conditions:

$$P\Big(\bar{X} < c_1|\mu = 0.1\Big) = 0.005 \qquad \text{and} \qquad P\Big(\bar{X} > c_2|\mu = 0.1\Big) = 0.005$$

To compute $c_1$ and $c_2$ we need to find the distribution of the random variable $\bar{X}$. The random sample is normal, so we know that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$. Under the null hypothesis $\bar{X} \sim N(0.1, \frac{1^2}{10})$, and therefore,

$$P\Bigg(\frac{\sqrt{10}(\bar{X}-0.1)}{1} < \frac{\sqrt{10}(c_1-0.1)}{1}\Bigg) = 0.005 \quad \text{and} \quad P\Bigg(\frac{\sqrt{10}(\bar{X}-0.1)}{1} > \frac{\sqrt{10}(c_2-0.1)}{1}\Bigg) = 0.005$$

Since $\frac{\sqrt{10}(\bar{X}-0.1)}{1} = Z \sim N(0,1)$, we know that

$$P\Bigg(Z < \frac{\sqrt{10}(c_1-0.1)}{1}\Bigg) = 0.005 \quad \text{and} \quad P\Bigg(Z > \frac{\sqrt{10}(c_2-0.1)}{1}\Bigg) = 0.005$$

Since $P(Z < z_{0.005}) = 0.005$ and $P(Z > z_{0.995}) = 0.005$, where $z_{0.005} = -2.575$ and $z_{0.995} = 2.575$, we know that

$$\frac{\sqrt{10}(c_1-0.1)}{1} = -2.575 \quad \text{and} \quad \frac{\sqrt{10}(c_2-0.1)}{1} = 2.575$$

and that

$$c_1 = 0.1 + \frac{-2.575}{\sqrt{10}} = -0.714 \quad \text{and} \quad c_2 = 0.1 + \frac{2.575}{\sqrt{10}} = 0.914$$

• Note that to compute $c_2$ we follow the same steps used to compute $c$ in the One-sided Test done above (with $\alpha = 0.005$).

### 0.4.2 Result of the test

$\bar{x} = 0.54 > c_1 = -0.714$ and $\bar{x} = 0.54 < c_2 = 0.914$, so we cannot reject at 1% the null hypothesis that the population mean is 0.1 against the alternative hypothesis that the population mean is different than 0.1.

## 0.5 Generalized Likelihood Ratio Test (GLRT):

We will no solve this test, but we will just give the basic guidelines.

$$H_0: \quad \mu = 0$$
$$H_1: \quad \mu \neq 0$$

Decision Rule form: "Reject $H_0$ if $T > d$".

Where $T$ is the following statistic:

$$T = \frac{\sup_{\theta \in \Omega_0} L(\theta_1, ..., \theta_k | x_1, ..., x_n)}{\sup_{\theta \in \Omega} L(\theta_1, ..., \theta_k | x_1, ..., x_n)} = \frac{\sup_{\theta \in \Omega_0} f(\mathbf{x} | \theta \in \Omega_0)}{\sup_{\theta \in \Omega} f(\mathbf{x} | \theta \in \Omega)}$$

• We need to compute $T$ and $d$. For $T$, we need to find the maximum value of the likelihood function (given the realization of the sample) evaluating $\mu$ at all the possible values in the null and alternative set ($\forall \mu \in \Re$ in this case). This maximum value of the likelihood function is the number that goes in the denominator.

• We need to follow the same procedure to compute the value in the numerator, but we only need to evaluate the likelihood at all possible values of $\mu$ in the null space. Since the null space is singleton in this case, we have to evaluate the likelihood only at $\mu = 0$.

• We compute $d$ using the following condition, $P(T < d | \mu = 0) = \alpha$. As with the case of the LR-Test, the strategy to determine the distribution of $T$ depends on each case and many times it is be very hard to compute. We have an alternative when computing a GLRT, which applies only to the cases where we can assume that $n \to \infty$. In a large sample, we know that the limiting distribution of $-2lnT$ is (this result is not proved in this class):

$$-2lnT \overset{n \to \infty}{\sim} \chi^2_{(r)} \; ;$$

where $r$ is the # of free parameters in $\Omega$ minus the # of free parameters in $\Omega_0$. [1]

• Reject $H_0$ if $-2lnT > \chi^2_{(r),\alpha}$.

[1] The technical result says that the distribution is a $\chi^2_{(r)}$ with degrees of freedom $r = \dim\Omega - \dim\Omega_0$.