14.30 PROBLEM SET 3

Due: Tuesday, March 8, 2006, by 4:30 p.m.

Note: The first four problems are required, and the remaining two are practice problems. If you choose to do the practice problems now, you will receive feedback from the grader. Alternatively, you may use them later in the course to study for exams. Credit may not be awarded for solutions that do not use the methods discussed in this class.

Problem 1

Let X be a continuous random variable with pdf $f_X(x)$, and let a, b and c be constants.

a. Prove that for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist, $E(ag_1(X) + bg_2(X) + c) = aE[g_1(X)] + bE[g_2(X)] + c$.

b. Can you also state that: $E(g_1(X)) = g_1(E[X])$?

c. What can you say about $Var(g_1(X) + g_2(X))$?

Problem 2

Suppose that a random variable X_1 is distributed uniform $[0,1]$, X_2 is distributed uniform $[0,2]$ and X_3 is distributed uniform $[0,3]$. Assume that they are all independent.

a. Calculate $E[(X_1 - 2X_2 + X_3)]$.

b. Calculate $E\left[(X_1 - 2X_2 + X_3)^2\right]$.

c. Use your result from parts a. and b. to calculate $Var[(X_1 - 2X_2 + X_3)]$.

Problem 3

Let X and Y be two random variables, where $f_{X,Y}(-1,1) = \frac{1}{2}$, $f_{X,Y}(0,2) = \frac{1}{4}$, $f_{X,Y}(2,1) = \frac{1}{4}$, and $f_{X,Y}(x,y) = 0$ otherwise.

a. What is the correlation between X and Y?

b. Are X and Y independent?

c. What do you conclude about the relationship between correlation and independence?

Problem 4

Suppose that you are considering undertaking a project where the revenue, X_1, has a distribution: $f_{X_1}(x_1) = \begin{cases} \frac{3}{4} \text{ if } x_1 = 0 \\ \frac{1}{4} \text{ if } x_1 = 8 \\ 0 \text{ otherwise} \end{cases}$. The project cost, Y_1, is distributed: $f_{Y_1}(y_1) = \begin{cases} \frac{1}{2} \text{ if } y_1 \in \left\{1, \frac{3}{2}\right\} \\ 0 \text{ otherwise} \end{cases}$. Assume that the revenue and the cost are independent.

a. What is the expected profit (revenue minus cost) of the project?

b. What is the probability that the project makes a positive profit?

c. What is the variance of the profit of the project?

d. Now assume instead that X_1 and Y_1 are not independent, but instead that $f_{X_1|Y_1=\frac{3}{2}}\left(x_1|Y_1 = \frac{3}{2}\right) = \begin{cases} \frac{5}{8} \text{ if } x_1 = 0 \\ \frac{3}{8} \text{ if } x_1 = 8 \\ 0 \text{ otherwise} \end{cases}$. Now what is the expected value and variance of the profit?

e. Assume again that the cost and revenue are independent, and suppose that now a friend can undertake a different project where the revenue (X_2) is distributed identically to X_1 and the the cost (Y_2) is distributed identically to Y_1. All the variables X_1, Y_1, X_2, Y_2 are independent. Before you undertake either project your friend suggests that you equally share the revenue and costs of both projects. What is the expected profit to accepting the suggestion relative to not taking up any project? What is the probability that you make a positive profit if you accept your friend's suggestion? And what is the variance of the your share of the combined profit?

Problem 5

A student is enrolled in n classes. For each class, the probability that the student has an assignment due on a given day is independent, and equal to 0.7. The total time (T) it takes a student to compete k assignments is distributed according to the pdf $f_T(t) = \frac{1}{2k}e^{-\frac{t}{2k}}$ for $k > 0$. The student spends no time on homework if nothing is due.

a. If the student is enrolled in 4 classes, what is the expected value of k? What is the variance of k?

b. If $k = 2$, what is the expected value of T?

c. If the student is enrolled in 4 classes, what is the expected value of T? What is the variance of T?

Problem 6

Suppose that there are two kinds of stock, and that each of them currently sells for the same price in the stock market. The return on one share of the first stock is a random variable X with $E(X) = \mu$ and $Var(X) = \sigma^2$. The return on one share of the second stock (Y) also has $E(Y) = \mu$ and $Var(Y) = \sigma^2$. Consider the following investment plans:

Plan 1: Buy 10 shares of the first stock.

Plan 2: Buy 5 shares of each stock.

a. Compute the average return for each of the two plans and compare.

b. Suppose that the two plans are independent. What is the variance on the return to Plan 2?

c. Now suppose that the two plans are correlated with $\rho = \frac{1}{2}$. What is the variance on the return to Plan 2?

d. Now suppose that the two plans are correlated, but ρ is unknown. If you are risk-averse (i.e. you prefer less variance for a given expected return), which plan do you prefer? Why?