14.30 PROBLEM SET 2 - SUGGESTED ANSWERS

Problem 1

a. There are $\binom{12}{7}\binom{5}{3}$ different ways for the team to accumulate a 7-3-2 record, and the probability of each of these outcomes is $\left(\frac{1}{3}\right)^7\left(\frac{1}{2}\right)^3\left(\frac{1}{6}\right)^2$. So the probability of this record is $\binom{12}{7}\binom{5}{3}\left(\frac{1}{3}\right)^7\left(\frac{1}{2}\right)^3\left(\frac{1}{6}\right)^2 = \frac{55}{4374}$.

b. We can do a calculation like the one above for various records after four games. Then we will sum the records with equivalent point values to get the pmf.

Record 4-0-0: $\binom{4}{4}\left(\frac{1}{3}\right)^4 = \frac{1}{81}$, Scores 12 points.
Record 3-1-0: $\binom{4}{3}\binom{1}{1}\left(\frac{1}{3}\right)^3\left(\frac{1}{2}\right) = \frac{2}{27}$, Scores 9 points.
Record 3-0-1: $\binom{4}{3}\binom{1}{1}\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right) = \frac{2}{81}$, Scores 10 points.
Record 2-2-0: $\binom{4}{2}\binom{2}{2}\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)^2 = \frac{1}{6}$, Scores 6 points.
Record 2-1-1: $\binom{4}{2}\binom{2}{1}\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{1}{9}$, Scores 7 points.
Record 2-0-2: $\binom{4}{2}\binom{2}{2}\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right)^2 = \frac{1}{54}$, Scores 8 points.
Record 1-3-0: $\binom{4}{1}\binom{3}{3}\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^3 = \frac{1}{6}$, Scores 3 points.
Record 1-2-1: $\binom{4}{1}\binom{3}{2}\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right) = \frac{1}{6}$, Scores 4 points.
Record 1-1-2: $\binom{4}{1}\binom{3}{2}\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)^2 = \frac{1}{18}$, Scores 5 points.
Record 1-0-3: $\binom{4}{1}\binom{3}{3}\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)^3 = \frac{1}{162}$, Scores 6 points.
Record 0-4-0: $\binom{4}{4}\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, Scores 0 points.
Record 0-3-1: $\binom{4}{3}\binom{1}{1}\left(\frac{1}{2}\right)^3\left(\frac{1}{6}\right) = \frac{1}{12}$, Scores 1 point.
Record 0-2-2: $\binom{4}{2}\binom{2}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right)^2 = \frac{1}{24}$, Scores 2 points.
Record 0-1-3: $\binom{4}{1}\binom{3}{3}\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)^3 = \frac{1}{108}$, Scores 3 points.
Record 0-0-4: $\binom{4}{4}\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$, Scores 4 points.

So the pmf is $f(x) = \begin{cases} \frac{1}{16} \text{ if } x = 0 \\ \frac{1}{12} \text{ if } x = 1 \\ \frac{1}{24} \text{ if } x = 2 \\ \frac{19}{108} \text{ if } x = 3 \\ \frac{217}{1296} \text{ if } x = 4 \\ \frac{1}{18} \text{ if } x = 5 \\ \frac{14}{81} \text{ if } x = 6 \\ \frac{1}{9} \text{ if } x = 7 \\ \frac{1}{54} \text{ if } x = 8 \\ \frac{2}{27} \text{ if } x = 9 \\ \frac{2}{81} \text{ if } x = 10 \\ \frac{1}{81} \text{ if } x = 12 \\ 0 \text{ otherwise} \end{cases}$

For graph, see attached sheet.

c. $F(x) = \sum_{x_i \leq x} f(x_i)$, so we have the following cdf:

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{1}{16} \text{ if } 0 \leq x < 1 \\ \frac{7}{48} \text{ if } 1 \leq x < 2 \\ \frac{3}{16} \text{ if } 2 \leq x < 3 \\ \frac{157}{432} \text{ if } 3 \leq x < 4 \\ \frac{43}{81} \text{ if } 4 \leq x < 5 \\ \frac{95}{162} \text{ if } 5 \leq x < 6 \\ \frac{41}{54} \text{ if } 6 \leq x < 7 \\ \frac{47}{54} \text{ if } 7 \leq x < 8 \\ \frac{8}{9} \text{ if } 8 \leq x < 9 \\ \frac{74}{81} \text{ if } 9 \leq x < 10 \\ \frac{80}{81} \text{ if } 10 \leq x < 12 \\ 1 \text{ if } x \geq 12\frac{1}{81} \end{cases}$$

And the probability of having at least five points is $1 - F(4) = \frac{38}{81}$. Again, see the graph on the attached sheet.

Problem 2

a. We can calculate c by integrating the pdf: $1 = \lim_{x\to\infty} F_X(x) = \int_0^\infty f_X(x)\,dx = \int_0^\infty ce^{-\frac{x}{\beta}}dx$. Using the substitution: $u = \frac{x}{\beta}, du = \frac{1}{\beta}dx$ we get: $1 = \int_0^\infty c\beta e^{-u}du = c\beta\left[-e^{-u}\right]_0^\infty = c\beta\left(-(0-1)\right) = c\beta$. Therefore $c = \frac{1}{\beta}$ and $f_X(x) = \frac{1}{\beta}e^{-\frac{x}{\beta}}$ (exponential distribution).

b. $P(X > t) = \int_t^\infty f_X(x)\,dx = \int_t^\infty \frac{1}{\beta}e^{-\frac{x}{\beta}}dx$. Using the same substitution as before: $P(X > t) = \int_{t/\beta}^\infty \frac{1}{\beta}\beta e^{-u}du = \left[-e^{-u}\right]_{t/\beta}^\infty = e^{-t/\beta}$

c. $P(X > t+s|X > s) = \frac{P(X>t+s, X>s)}{P(X>s)} = \frac{P(X>t+s)}{P(X>s)}$ because $t > 0$. Using the answer to the previous question: $P(X > t+s) = e^{-(t+s)/\beta}$, $P(X > s) = e^{-s/\beta}$ so: $P(X > t+s|X > s) = \frac{e^{-(t+s)/\beta}}{e^{-s/\beta}} = e^{-t/\beta}$.

d. You are neither more nor less concerned that the batteries will die. The exponential distribution has a "memoryless" property - the probability that an event will occur is independent of the time that has elapsed.

Problem 3

a. By the property of the pdf, it must be true that

$$1 = \int_0^1 \int_x^1 kx^3 y dy dx = \int_0^1 \int_0^y kx^3 y dx dy = \frac{1}{24}k.$$

(draw a graph of x and y to determine the area of interest, which is a triangle). Hence, $k = 24$.

b. Paying careful attantion to the bounds of integration, solve for marginal pdf of x :

$$\int_x^1 24x^3 y dy = [12x^3y^2]_x^1 = 12x^3 - 12x^5 = 12x^3(1-x^2).$$

Therefore, the marginal pdf is defined as:

$$f_X(x) = \begin{cases} 12x^3(1-x^2) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

c. To get the cdf of x at $1/2$, note that

$$F_X(\tfrac{1}{2}) = P(x \leq 1/2) = \int_{-\infty}^{1/2} f_X(x)dx = \int_0^{1/2} 12x^3(1-x^2)dx = \frac{5}{32}$$

d. The conditional pdf of y is defined as

$$f(y \mid x) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)} & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} \frac{2y}{(1-x^2)} & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

To determine whether X and Y are independent, we need to find the marginal distribution of Y:

$$f_Y(y) = \int_y 24x^3 y dx = [6x^4 y]_{x=0}^{x=y} = 6y^5 \text{ (if } 0 < y < 1).$$

Because this is not equal to the conditional pdf of Y, we know that X and Y are not independent.

e. We find the probability integrating the pdf over relevant range. A graph will show us that the area is a triangle bounded by the y-axis, the line $y = x$, and the line $y = 1 - x$. So we have

$$\begin{aligned}\Pr(X + Y < 1) &= \int_0^{\frac{1}{2}} \int_x^{1-x} 24x^3 y dy dx \\ &= \int_0^{\frac{1}{2}} 12x^3(1-2x)dx \\ &= \frac{3}{80}\end{aligned}$$

Problem 4

a. If the pdf of X at point x is proportional to x then: $f_X(x) = \begin{cases} kx \text{ if } x \in [0,1] \\ 0 \text{ otherwise} \end{cases}$. We can find k by integration: $1 = \int_0^1 (kx)\,dx = \frac{k}{2}\left[x^2\right]_0^1 = \frac{k}{2}$ so $k = 2$. Therefore the pdf is: $f_X(x) = \begin{cases} 2x \text{ if } x \in [0,1] \\ 0 \text{ otherwise} \end{cases}$. We can calculate the cdf by integrating the pdf: $F_X(x) = \int_0^x (2u)\,du = \left[u^2\right]_0^x = x^2$. Hence: $F_X(x) = \begin{cases} 0 \text{ if } x < 0 \\ x^2 \text{ if } x \in [0,1] \\ 1 \text{ if } x > 1 \end{cases}$

b. The cdf is: $F_X(x) = \begin{cases} 0 \text{ if } x < 0 \\ kx \text{ if } x \in [0,1] \\ 1 \text{ if } x > 1 \end{cases}$. We can solve for k by using the right-continuity of F for $x = 1$, that is: $1 = \lim_{x \downarrow 1} F_X(x) = F_X(1) = k$, so $k = 1$ and therefore: $F_X(x) = \begin{cases} 0 \text{ if } x < 0 \\ x \text{ if } x \in [0,1] \\ 1 \text{ if } x > 1 \end{cases}$. We can find the pdf by differentiating the cdf over the interval $[0,1]$: $f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}x = 1$. Therefore: $f_X(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 0 \text{ otherwise} \end{cases}$

Problem 5

First, this is a hard problem, and part d should have come before b, so sorry about that. Also, I apologize for the typo in part d (it should ask for $f_{Y|Z}(y \mid z)$).

a. Y has a Binomial distribution: $f_Y(y) = P(Y = y) = \binom{n}{y} p^y (1-p)^y$ for $y = 0, 1, 2, ..., n$ (and 0 otherwise).

b. First we find the marginal distribution of Z: $f_Z(z) = \begin{cases} p^z(1-p) \text{ if } z = 0, 1, 2, ... \\ 0 \text{ otherwise} \end{cases}$ (geometric distribution). Remember that it is not always possible to derive the joint distribution from the marginal distributions. However, in this case we can also figure out the conditional distribution of y, and use that to find the joint distribution. As long as $y \geq z$, $Y \mid Z$ also has the binomial form, but we use $n - (z+1)$ as the number of trials remaining, and $y - z$ is the number of additional successes needed. So, given that n, y, and z are whole numbers (and $n > y$), we have

$$f_{Y|Z}(y \mid z) = \begin{cases} 1 & \text{if } y = n \leq z \\ \binom{n-(z+1)}{y-z} p^{y-z} (1-p)^{n-1-y} & \text{if } z \leq y \leq n \text{ and } z \neq n \\ 0 & \text{elsewhere} \end{cases}$$

Now we can use our formula for the conditional distribution to get the joint distribution:

$$f_{Y|Z}(y \mid z) = \frac{f_{Y,Z}(y,z)}{f_Z(z)}$$

$$f_{Y,Z}(y,z) = \begin{cases} p^z (1-p) & \text{if } y = n \leq z \\ \binom{n-(z+1)}{y-z} p^{y} (1-p)^{n-y} & \text{if } z \leq y \leq n \text{ and } z \neq n \\ 0 & \text{elsewhere} \end{cases}$$

c. The marginal pdf of Y will be the same as the distribution for Y found in part a. Proving that the summation of $f_{Y,Z}(y,z)$ over all z is equal to this is rather tricky, and I didn't intend for you to do so.

d. We found this in part b.

e. No, because $f_{Y,Z}(y,z) \neq f_Y(y)$.

Problem 6

Again we encounter a geometric distribution (note that we do not count the week that Eliza finally gives her presentation):

$$f_X(x) = \begin{cases} p(1-p)^x & \text{if } x = 0, 1, 2, ... \\ 0 & \text{otherwise} \end{cases}$$

But we need to determine what p is for Eliza, or what is the probability that she will be allowed to present in a given week.

$$p = \Pr(\text{sum} = 11) + \Pr(\text{sum} = 12) = \binom{3}{2}\binom{1}{1}\frac{1}{4^3} + \binom{3}{3}\frac{1}{4^3} = \frac{1}{16}$$