14.30 PROBLEM SET 1 - SUGGESTED ANSWERS

Problem 1

a. For any A and B, we can create two sets that are disjoint and exhaustive on B: $A \cap B$ and $A^C \cap B$. Thus

$$\Pr(B) = \Pr(A \cap B) + \Pr\left(A^C \cap B\right)$$

Because $A \subset B$, we know that $A = A \cap B$. So we have

$$\begin{aligned} \Pr(B) &= \Pr(A) + \Pr\left(A^C \cap B\right) \\ \Pr(B) - \Pr(A) &= \Pr\left(A^C \cap B\right) \\ \Pr(B) - \Pr(A) &\geq 0 \\ \Pr(B) &\geq \Pr(A) \end{aligned}$$

Becase $\Pr\left(A^C \cap B\right) \geq 0$.

b. We know that in general,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

so rearranging terms, we have that

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$$

And because $\Pr(A \cup B) \leq 1$,

$$\Pr(A \cap B) \geq \Pr(A) + \Pr(B) - 1$$

Problem 2

a. $\binom{15}{3} = \frac{15!}{3!12!} = \frac{15 \times 14 \times 13}{3 \times 2} = 455$

b. Six students have already been chosen, so by the third jump, there are only nine students left to chose from. So we have

$$\binom{9}{3} = \frac{9!}{3!6!} = \frac{9 \times 8 \times 7}{3 \times 2} = 84$$

c. We want the unconditional probability of an individual (let's call her Sarah) being in the third group. Since three people out of fifteen will be in the third group, Sarah will be in the third group with probability 0.2 (which is the same as the probability of being in any other group).

For the purpose of illustration, I will also show how you could calculate this probability using our definition of conditional probabilities:

$$\Pr(3rd \mid Not\ 1st\ two) = \frac{\Pr(3rd \cap Not\ 1st\ two)}{\Pr(Not\ 1st\ two)}$$

We can rearrange this to get a formula for the probability that we want.

$$\Pr(3rd \cap Not\ 1st\ two) = \Pr(3rd \mid Not\ 1st\ two)\Pr(Not\ 1st\ two)$$
$$\Pr(3rd \cap Not\ 1st\ two) = \Pr(3rd \mid Not\ 1st\ two)\Pr(Not\ 2nd \mid Not\ 1st)\Pr(Not\ 1st)$$

We can calculate each of these probabilities using the rule $\Pr(A) = \frac{N(A)}{N}$. We start with $\Pr(Not\ 1st)$. We know from part a that $N = 455$. To get $N(Not\ 1st)$, we must calculate the number of ways that the first group can be formed without choosing Sarah, which is

$$\binom{14}{3} = \frac{14!}{3!11!} = \frac{14 \times 13 \times 12}{3 \times 2} = 364$$

so $\Pr(Not\ first) = \frac{364}{455} = 0.8$
Then we need

$$\Pr(Not\ 2nd \mid Not\ 1st) = \frac{\binom{11}{3}}{\binom{12}{3}} = \frac{\frac{11!}{3!8!}}{\frac{12!}{3!9!}} = \frac{9}{12} = 0.75$$

And finally

$$\Pr(3rd \mid Not\ 1st\ two) = \frac{\binom{8}{2}}{84} = \frac{\frac{8\times7}{2}}{84} = 0.\overline{3}$$

So

$$\Pr(3rd \cap Not\ 1st\ two) = \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = 0.2$$

d. We sum the number of ways the group could be chosen excluding the couple and including the couple:

$$\binom{13}{3} + \binom{13}{1} = \frac{13 \times 12 \times 11}{3 \times 2} + 13 = 299$$

e. $\Pr(couple) = \frac{N(couple)}{N} = \frac{13}{299}$

f. From our answer to part d, we subtract the number of combinations that have three of these five together:

$$\binom{13}{3} + \binom{13}{1} - \binom{5}{3} = 299 - \frac{5 \times 4}{2} = 289$$

g. $\Pr(couple) = \frac{N(couple)}{N} = \frac{13}{289}$

Problem 3

a. We want to find $\Pr(C \mid \text{positive})$, and we know the probabilities of a positive test for patients with and without cancer, as well as the baseline probability, so we can use Bayes rule:

$$\Pr(C \mid \text{positive}) = \frac{\Pr(\text{positive} \mid C)\Pr(C)}{\Pr(\text{positive} \mid C)\Pr(C) + \Pr(\text{positive} \mid C^C)\Pr(C^C)}$$

We substitute in to get

$$\Pr(C \mid \text{positive}) = \frac{.8 \times .05}{.8 \times .05 + .1 \times .95} = 0.296$$

b. We use the same method.

$$\begin{aligned}\Pr\left(C^C \mid \text{negative}\right) &= \frac{\Pr\left(\text{negative} \mid C^C\right)\Pr\left(C^C\right)}{\Pr(\text{negative} \mid C)\Pr(C) + \Pr(\text{negative} \mid C^C)\Pr(C^C)} \\ \Pr\left(C^C \mid \text{negative}\right) &= \frac{.9 \times .95}{.2 \times .05 + .9 \times .95} = 0.988\end{aligned}$$

Problem 4

Galileo noted that different combinations of spots may occur with differing probabilities. Combinations with three different numbers (1,3,6) can occur 3! different ways, combinations with two different numbers (1,4,4) can occur 3 different ways, and combinations with only one number (3,3,3) can occur in only one way. Overall, there are 6^3 permutations of dice outcomes. So,

$$\begin{aligned}\Pr(9) &= \frac{3 \times 3! + 2 \times 3 + 1}{6^3} = \frac{25}{216} \\ \Pr(10) &= \frac{3 \times 3! + 3 \times 3}{6^3} = \frac{27}{216}\end{aligned}$$

Problem 5

a. We start by calcualting the total number of possible hands, which is $\binom{52}{5} = 2598960$. Next we want the number of hands containing five cards of the same suit: $4 \times \binom{13}{5} = 5148$. So

$$\Pr(\textit{all same suit}) = \frac{5148}{2598960} = 0.00198$$

b. There are $\binom{13}{2}$ ways to choose the denominations of the two pairs, and then $\binom{4}{2}$ ways to choose the suits of the cards in each pair. Finally, there are 11 denominations left for the final card, in any of the four suits. So the probability calculation is

$$\Pr(\textit{two pair}) = \frac{N(\textit{two pair})}{N} = \frac{\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times 11 \times 4}{\binom{52}{5}} = \frac{123552}{2598960} = 0.0475$$

c. There are $\binom{13}{1}$ denominations of the triple, and $\binom{4}{3}$ choices of suits. For the remaining two cards, there are $\binom{12}{2}$ denominations and four suits for each card.

$$\Pr(triple) = \frac{N(triple)}{N} = \frac{\binom{13}{1} \times \binom{4}{3} \times \binom{12}{2} \times 4^2}{\binom{52}{5}} = \frac{54912}{2598960} = 0.0211$$

d. Again, we have $\binom{13}{1}$ denominations of the set, but only one combination of suits. The remaining card can have $\binom{12}{1}$ denominations in any of the four suits.

$$\Pr(four\ of\ a\ kind) = \frac{N(four\ of\ a\ kind)}{N} = \frac{\binom{13}{1} \times \binom{12}{1} \times 4}{\binom{52}{5}} = \frac{624}{2598960} = 0.00024$$

e. Now we only have one choice of denominations and one combination of suits. The remaining card still has $\binom{12}{1}$ denominations in any of the four suits.

$$\Pr(four\ Aces) = \frac{N(four\ Aces)}{N} = \frac{\binom{12}{1} \times 4}{\binom{52}{5}} = \frac{48}{2598960} = 0.0000185$$

Problem 6

a. Calculating the probability that a new policy holder gets in an accident in the first year uses the law of total probability.

$$\begin{aligned} \Pr(A) &= \Pr(A \mid S)\Pr(S) + \Pr(A \mid U)\Pr(U) \\ &= 0.1 \times 0.5 + 0.4 \times 0.5 \\ &= 0.25 \end{aligned}$$

b. Now we will use Bayes rule:

$$\begin{aligned} \Pr(U \mid A^c) &= \frac{\Pr(A^c \mid U)\Pr(U)}{\Pr(A^c \mid S)\Pr(S) + \Pr(A^c \mid U)\Pr(U)} \\ &= \frac{0.6 \times 0.5}{0.9 \times 0.5 + 0.6 \times 0.5} \\ &= 0.4 \end{aligned}$$

c. Again we start by using Bayes rule to get the probability that a driver is unsafe after an accident in the first year:

$$\begin{aligned} \Pr(U \mid A) &= \frac{\Pr(A \mid U)\Pr(U)}{\Pr(A \mid S)\Pr(S) + \Pr(A \mid U)\Pr(U)} \\ &= \frac{0.4 \times 0.5}{0.1 \times 0.5 + 0.4 \times 0.5} \\ &= 0.8 \end{aligned}$$

Now we repeat the process for the second year, using 0.8 as our baseline probability that the driver is unsafe.

$$\begin{aligned}
\Pr(U \mid A) &= \frac{\Pr(A \mid U)\Pr(U)}{\Pr(A \mid S)\Pr(S) + \Pr(A \mid U)\Pr(U)} \\
&= \frac{0.4 \times 0.8}{0.1 \times 0.2 + 0.4 \times 0.8} \\
&= 0.94
\end{aligned}$$

So after only the second year with an accident, the insurance company is more than ninety percent sure that the driver is unsafe.