14.30 PROBLEM SET 5 - SUGGESTED ANSWERS

Problem 1

The joint pdf of X and Y will be equal to the product of the marginal pdfs, since X and Y are independent.

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2+y^2\right)} \end{aligned}$$

The transformation into polar coordinates is

$$\begin{aligned} r^2 &= X^2 + Y^2 \\ \tan\theta &= \frac{Y}{X} \end{aligned}$$

with inverse transformations

$$\begin{aligned} X &= r\cos\theta \\ Y &= r\sin\theta \end{aligned}$$

This yeilds the following matrix of partial derivatives.

$$\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The determinant of this matrix, the Jacobian, is

$$\begin{aligned} J &= \cos\theta\,(r\cos\theta) - \sin\theta\,(-r\sin\theta) \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r\left(\cos^2\theta + \sin^2\theta\right) = r \end{aligned}$$

The transformations are unique, so we can use the 1-step method without modification.

$$f_{r\theta}(r,\theta) = r\frac{1}{2\pi} e^{-\frac{1}{2}r^2}$$

where r lies within $[0, \infty]$ and θ lies within $[0, 2\pi]$. Because the ranges are not dependent and the joint pdf is separable, r and θ are also independent.

Problem 2

a. For a single random variable: $P(X_i \leq 115) = P\left(\frac{X_i-\mu}{\sigma} \leq \frac{115-\mu}{\sigma}\right)$. Notice that $Z_i = \frac{X_i-\mu}{\sigma}$ is distributed standard normal ($Z_i \sim N(0,1)$) so:

$P(X_i \le 115) = P\left(Z_i \le \frac{115-100}{\sqrt{225}}\right) = P(Z_i \le 1)$. Using the Table you can find that this probability is approximately equal to: 0.8413. By independence: $P(X_1 \le 115, X_2 \le 115, X_3 \le 115, X_4 \le 115) =$
$P(X_1 \le 115) P(X_2 \le 115) P(X_3 \le 115) P(X_4 \le 115) = 0.8413^4 = 0.500\,96$.

b. $\overline{X}_n = \sum_{i=1}^n \frac{1}{n} X_i \sim N\left(\sum_{i=1}^n \frac{1}{n}\mu_i, \sum_{i=1}^n \left(\frac{1}{n}\right)^2 \sigma_i^2\right) = N\left(100, \frac{225}{n}\right) = N\left(100, \left(\frac{15}{\sqrt{n}}\right)^2\right)$, so: $\overline{X}_4 \sim N\left(100, \left(\frac{15}{2}\right)^2\right)$. Thus: $Z = \frac{\overline{X}_4 - 100}{\left(\frac{15}{2}\right)}$ is a standard normal random variable: $P\left(\overline{X}_4 < 115\right) = P\left(\frac{\overline{X}_4-100}{\left(\frac{15}{2}\right)} < \frac{115-100}{\left(\frac{15}{2}\right)}\right) = P(Z < 2) = 0.9772$.

c. $P\left(\left|\overline{X}_n - \mu\right| \le 5\right) = P\left(\left|\frac{\overline{X}_n-\mu}{\left(\frac{15}{\sqrt{n}}\right)}\right| \le \frac{5}{\left(\frac{15}{\sqrt{n}}\right)}\right) = P\left(\frac{-\sqrt[2]{n}}{3} \le Z \le \frac{\sqrt{n}}{3}\right) = 0.95$. From the table we know that: $P(Z \le 1.96) \simeq 0.975$ and using the symmetry of the normal distribution this implies that $P(-1.96 \le Z \le 1.96) \simeq 0.95$, so $\frac{\sqrt{n}}{3} = 1.96 \Rightarrow n = (1.96 \cdot 3)^2 = 34.574$. We want the smallest integer and it is $n_0 = 35$.

Problem 3

a. The number of heads (H) in 10 independent flips of a fair coin is distributed $Binomial\left(10, \frac{1}{2}\right)$. $P(0 \le H \le 4) = \sum_{k=0}^4 \binom{10}{k} (0.5)^k (0.5)^{10-k} = \sum_{k=0}^4 \binom{10}{k} (0.5)^{10} =$
$= (0.5)^{10} \left[\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4}\right] = \frac{386}{1024} = 0.376\,95$

b. Since H is binomial we can calculate its mean and variance: $E[H] = 10 \cdot (0.5) = 5, Var[H] = 10 \cdot (0.5)(1-0.5) = 2.5$. The approximation relies on the assumption that H is distributed similar to a normal random variable, so: $\frac{H-E[H]}{\sqrt{Var[H]}} = \frac{H-5}{\sqrt{2.5}} \simeq Z \sim N(0,1)$. Therefore: $P(0 \le H \le 4) = P\left(\frac{0-E[H]}{\sqrt{Var[H]}} \le \frac{H-E[H]}{\sqrt{Var[H]}} \le \frac{4-E[H]}{\sqrt{Var[H]}}\right) \simeq P\left(\frac{-5}{\sqrt{2.5}} \le Z \le \frac{-1}{\sqrt{2.5}}\right) \simeq P(-3.162 \le Z \le -0.632) = P(Z \le 3.162) - P(Z \le 0.632) \simeq 0.999 - 0.736 = 0.263$. Thus the approximation is not very accurate for $n = 10$.

c. Now $P(0 \le H \le 40) = P\left(\frac{0-E[H]}{\sqrt{Var[H]}} \le \frac{H-E[H]}{\sqrt{Var[H]}} \le \frac{40-E[H]}{\sqrt{Var[H]}}\right) \simeq P\left(\frac{-50}{\sqrt{25}} \le Z \le \frac{-10}{\sqrt{25}}\right) = P(-10 \le Z \le -2) = P(Z \le 10) - P(Z \le 2) \simeq 1 - 0.977 = 0.023$, which is quite close to the exact probability.

d. Exact calculation: $P(H = 6) = \binom{100}{6}\left(\frac{1}{20}\right)^6 \left(1 - \frac{1}{20}\right)^{100-6} = 0.15$. Approximation: as $n \to \infty, p \to 0, (np) \to \lambda$ the binomial distribution converges to the Poisson distribution with parameter λ. Since here $np =$

5 we can approximate the distribution with a Poisson distribution where $\lambda = 5$: $P(H = 40) \simeq \frac{e^{-\lambda}\lambda^6}{6!} = \frac{e^{-5}5^6}{6!} = 0.146$. Clearly, this is a good approximation.

Problem 4

First of all, to have valid pdfs, we must use $f_{X_i}(x) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}$. As always, sorry about the typo (I omitted the negative sign).

a. Because X_1 and X_2 are independent, the joint pdf is again the product of the marginal pdfs:

$$\begin{aligned} f_{X_1X_2}(x_1, x_2) &= f_{X_1}(x_1) f_{X_2}(x_2) \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} \\ &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)} \end{aligned}$$

b. We will use Y_1 for Y. We start with the transformations

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned}$$

which will yeild the following inverse transformations:

$$\begin{aligned} X_1 &= \frac{Y_1 + Y_2}{2} \\ X_2 &= \frac{Y_1 - Y_2}{2} \end{aligned}$$

Then the matrix of partial derivatives is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

So the Jacobian is $\left|\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right| = \frac{1}{2}$

The transformation is unique, so we can use the 1-step method without modification.

$$
\begin{aligned}
f_{Y_1Y_2}(y_1,y_2) &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\left(\frac{\frac{y_1+y_2}{2}-\mu_1}{\sigma_1}\right)^2+\left(\frac{\frac{y_1-y_2}{2}-\mu_2}{\sigma_2}\right)^2\right)}\left(-\frac{1}{2}\right) \\
&= \frac{1}{4\pi\sigma_1\sigma_2} e^{-\frac{1}{2\sigma_1^2\sigma_2^2}\left(\sigma_2^2\left(\frac{y_1+y_2}{2}-\mu_1\right)^2+\sigma_1^2\left(\frac{y_1-y_2}{2}-\mu_2\right)^2\right)} \\
&= \frac{1}{4\pi\sigma_1\sigma_2} e^{-\frac{1}{2\sigma_1^2\sigma_2^2}\left(\frac{\sigma_2^2}{4}\left(y_1^2+2y_1y_2+y_2^2-4\mu_1y_1-4\mu_1y_2+4\mu_1^2\right)+\frac{\sigma_1^2}{4}\left(y_1^2-2y_1y_2+y_2^2-4\mu_2y_1+4\mu_2y_2+4\mu_2^2\right)\right)} \\
&= \frac{1}{4\pi\sigma_1\sigma_2} e^{-\frac{\left(\sigma_1^2+\sigma_2^2\right)\left((y_1-(\mu_1+\mu_2))^2+(y_2-(\mu_1-\mu_2))^2\right)}{8\sigma_1^2\sigma_2^2}} \times \\
&\quad e^{-\frac{2y_1y_2-2\mu_1y_2-2\mu_1y_1+2\mu_2y_1-2\mu_2y_2+2\mu_1^2-2\mu_2^2}{8\sigma_1^2}-\frac{-2y_1y_2+2\mu_1y_2+2\mu_1y_1-2\mu_2y_1+2\mu_2y_2-2\mu_1^2-2\mu_2^2}{8\sigma_2^2}}
\end{aligned}
$$

To get the pdf of Y, we must integrate over Y_2.

$$
\begin{aligned}
f_Y(y_1) &= \int_{-\infty}^{\infty} \frac{1}{4\pi\sigma_1\sigma_2} e^{-\frac{\left(\sigma_1^2+\sigma_2^2\right)\left((y_1-(\mu_1+\mu_2))^2+(y_2-(\mu_1-\mu_2))^2\right)}{8\sigma_1^2\sigma_2^2}} \times \\
&\quad e^{-\frac{2y_1y_2-2\mu_1y_2-2\mu_1y_1+2\mu_2y_1-2\mu_2y_2+2\mu_1^2-2\mu_2^2}{8\sigma_1^2}-\frac{-2y_1y_2+2\mu_1y_2+2\mu_1y_1-2\mu_2y_1+2\mu_2y_2-2\mu_1^2-2\mu_2^2}{8\sigma_2^2}} dy_2 \\
&= \frac{1}{4\pi\sigma_1\sigma_2} e^{-\frac{\left(\sigma_1^2+\sigma_2^2\right)(y_1-(\mu_1+\mu_2))^2}{8\sigma_1^2\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\frac{\left(\sigma_1^2+\sigma_2^2\right)(y_2-(\mu_1-\mu_2))^2}{8\sigma_1^2\sigma_2^2}} \times \\
&\quad e^{-\frac{2y_1y_2-2\mu_1y_2-2\mu_1y_1+2\mu_2y_1-2\mu_2y_2+2\mu_1^2-2\mu_2^2}{8\sigma_1^2}-\frac{-2y_1y_2+2\mu_1y_2+2\mu_1y_1-2\mu_2y_1+2\mu_2y_2-2\mu_1^2-2\mu_2^2}{8\sigma_2^2}} dy_2
\end{aligned}
$$

which has no closed form, in general. By other methods, it can be proved that $Y\tilde{}N\left(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2\right)$. We can show this here if we let $\sigma_1=\sigma_2=\sigma$.

$$
\begin{aligned}
f_Y(y_1) &= \frac{1}{4\pi\sigma^2} e^{-\frac{\left(2\sigma^2\right)(y_1-(\mu_1+\mu_2))^2}{8\sigma^4}} \int_{-\infty}^{\infty} e^{-\frac{\left(2\sigma^2\right)(y_2-(\mu_1-\mu_2))^2}{8\sigma^4}} \times \\
&\quad e^{-\frac{2y_1y_2-2\mu_1y_2-2\mu_1y_1+2\mu_2y_1-2\mu_2y_2+2\mu_1^2-2\mu_2^2}{8\sigma^2}-\frac{-2y_1y_2+2\mu_1y_2+2\mu_1y_1-2\mu_2y_1+2\mu_2y_2-2\mu_1^2-2\mu_2^2}{8\sigma^2}} dy_2 \\
&= \frac{1}{\sqrt{\pi}2\sigma} e^{-\frac{(y_1-(\mu_1+\mu_2))^2}{4\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}2\sigma} e^{-\frac{\left(2\sigma^2\right)(y_2-(\mu_1-\mu_2))^2}{8\sigma^4}} dy_2 \\
&= \frac{1}{\sqrt{\pi}2\sigma} e^{-\frac{(y_1-(\mu_1+\mu_2))^2}{4\sigma^2}}
\end{aligned}
$$

because $\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}2\sigma} e^{-\frac{\left(2\sigma^2\right)(y_2-(\mu_1-\mu_2))^2}{8\sigma^4}} dy_2$ is a standard normal, and must integrate to 1.

c. $E(Y) = E(X_1 + X_2) = \mu_1 + \mu_2$, since X_1 and X_2 are independent, normally distributed randome variables. Similarly, $V(Y) = V(X_1 + X_2) = \sigma_1^2 + \sigma_2^2$ (since X_1 and X_2 are independent, their covariance is zero).

Problem 5

a. X is distributed χ^2 with p degrees of freedom, so its pdf is

$$f(x) = \frac{1}{\Gamma\left(\frac{p}{2}\right) 2^{\frac{p}{2}}} x^{\frac{p}{2}-1} e^{-\frac{x}{2}}$$

A gamma distribution for a random variable Y is of the form

$$f(y) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-\frac{y}{\beta}}$$

You can see that if we let $\alpha = \frac{p}{2}$ and $\beta = 2$, X has a gamma distribution.

b. We learned in class that the square of a standard normal random variable has a χ^2 distribution with one degree of freedom. Thus $\left(\frac{y-\mu}{\sigma}\right)^2 \,\tilde{}\, \chi^2_{(1)}$. In addition, we learned that the sum of two independent χ^2 variables will also have a χ^2 distribution, with degrees of freedom equal to the sum of the degrees of freedom of initial random variables. Because Y and X are independent, Y^2 and X will also be independent, and we can apply this property to conclude that $\left(\frac{y-\mu}{\sigma}\right)^2 + X \,\tilde{}\, \chi^2_{(p+1)}$.

c. If $p = 4$, we use the fourth row of the table given in class, and look for the column corresponding to $\alpha = 0.05$. We can see that $A = 9.488$.