14.30 PROBLEM SET 5

Due: Tuesday, April 4, by 4:30 p.m.

Note: The first three problems are required, and the remaining two are practice problems. If you choose to do the practice problems now, you will receive feedback from the grader. Alternatively, you may use them later in the course to study for exams. Credit may not be awarded for solutions that do not use methods discussed in class.

Problem 1

Suppose that X and Y represent coordinates on the Cartesian plane, where X and Y both have a standard normal distribution and are independent. Find the joint pdf of the polar coordinates r and θ. (Remember that $r^2 = X^2 + Y^2$ and $tan\ \theta = \frac{Y}{X}$.) Are r and θ independent?

Problem 2

Suppose that n random variables X_i are independent and identically distributed as Normal: $X_i \sim N(\mu = 100, \sigma^2 = 225), i = 1, 2, ..., n$.

a. Consider the case where $n = 4$. What is the probability that *none* of the four random variables (X_1, X_2, X_3, X_4) is greater than 115?

b. Define the average of the n random variables as: $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. What is the probability that $\overline{X}_4$ is less than 115?

c. What is the *smallest integer* sample size n_0 that one would need in order to ensure that $P\left(\left|\overline{X}_{n_0} - \mu\right| \leq 5\right) \geq 0.95$?

Problem 3

Assume the coin flips below are independent flips of a fair coin.

a. Calculate the *exact* probability that the number of heads in ten flips is less than or equal to 4.

b. Use the *normal approximation* to the binomial distribution to approximate the probability you found in part a. and compare your answers in terms of accuracy.

c. Now assume that you flip the coin 100 times. The exact probability (up to three digits after the decimal point) that the number of heads is less than or equal to 40 is 0.028. What is the approximated probability (using the *normal* distribution) that the number of heads is less than or equal to 40?

d. Suppose we now begin to use a different coin such that the flips are still independent but the probability of heads in each toss is now $p = \frac{1}{20}$. Suppose you flip this coin $n = 100$ times. Compute the exact probability that $H = 6$ and compare it with an *approximation based on the Poisson distribution.*

Problem 4

Let X_1 and X_2 be normally distributed random variables, so that

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{\frac{(x-\mu_i)^2}{2\sigma^2}}$$

for $i = 1, 2$, and suppose that X_1 and X_2 are independent.

a. Find the joint pdf of X_1 and X_2.

b. Define $Y = X_1 + X_2$. Find the pdf of Y using the 1-step method (Hint: use $Y_2 = X_1 - X_2$ as an "ancilliary variable").

c. Find the mean and variance of Y.

Problem 5

Suppose that X is distributed chi-squared with p degrees of freedom, where p is a positive integer.

a. Write down the pdf of X. Show that X can also be characterized as having a gamma distribution. What are the parameters of the gamma distribution?

b. Now suppose that Y is distributed normally with mean μ and variance σ^2, and that Y and X are independent. Let $Z = \left(\frac{Y-\mu}{\sigma}\right)^2 + X$. What is the distribution of Z?

c. Suppose that $p = 4$. Use the appropriate table to find the value A such that $\Pr(Z > A) = 0.05$