

14.384 Time Series Analysis, Fall 2008
 Recitation by Paul Schrimpf
 Supplementary to lectures given by Anna Mikusheva
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Recitation 11

Filtering

Kalman filtering is a fancy name for a simple idea. Many people think that Kalman filtering is difficult or complicated because they think of it in terms of an updating algorithm with many steps. In fact, to understand Kalman filtering, it's best to forget the algorithm and just remember what it's for. Knowing what Kalman filtering is for, you should be able to derive the entire algorithm in an hour or so. The purpose of Kalman filtering is to compute the likelihood of a state-space model. Suppose we have some model with parameters θ , and we want to compute the likelihood of a sequence, $\{y_t\}$. Just as when we derived the likelihood for a VAR, we can break the joint likelihood into a product of marginal likelihoods:

$$\begin{aligned} f(y_1, \dots, y_T; \theta) &= f(y_T | y_{T-1}, \dots, y_1; \theta) f(y_{T-1} | y_{T-2}, \dots, y_1; \theta) \dots f(y_1; \theta) \\ &= \prod_{t=1}^T f(y_t | y_{t-1}, \dots, y_1; \theta) \end{aligned}$$

For a VAR, this very convenient because $f(y_t | y_{t-1}, \dots, y_1; \theta)$ has a simple form. For a state space model, $f(y_t | y_{t-1}, \dots, y_1; \theta)$ is not so simple because the model is written in terms of $f(y_t | \alpha_t; \theta)$ and $f(\alpha_t | \alpha_{t-1}; \theta)$ instead. Our goal is to write $f(y_t | y_{t-1}, \dots, y_1; \theta)$ in terms of these primitives. We can always write

$$f(y_t | y_{t-1}, \dots, y_1; \theta) = \int f(y_t | \alpha_t; \theta) f(\alpha_t | y_{t-1}, \dots, y_1; \theta) d\alpha_t \quad (1)$$

$f(y_t | \alpha_t; \theta)$ is a primitive of the model, so we're happy with it, but we need to simplify $f(\alpha_t | y_{t-1}, \dots, y_1; \theta)$. For $t = 1$, this is just $f(\alpha_1; \theta)$, which is equal to the stationary distribution of $f(\alpha_t | \alpha_{t-1}; \theta)$. For $t > 1$, we have

$$f(\alpha_t | y_{t-1}, \dots, y_1; \theta) = \int f(\alpha_t | \alpha_{t-1}; \theta) f(\alpha_{t-1} | y_{t-1}, \dots, y_1; \theta) d\alpha_{t-1} \quad (2)$$

Furthermore we know from Baye's rule that

$$f(\alpha_{t-1} | y_{t-1}, \dots, y_1; \theta) = \frac{f(y_{t-1} | \alpha_{t-1}; \theta) f(\alpha_{t-1} | y_{t-2}, \dots, y_1; \theta)}{f(y_{t-1} | y_{t-2}, \dots, y_1; \theta)} \quad (3)$$

Thus, we can compute the joint density as follows:

- $f(\alpha_1; \theta)$ to the invariant density of α .¹
- For $t = 1, \dots, T$
 1. Use (1) to compute $f(y_t | y_{t-1}, \dots, y_1; \theta)$
 2. Use (3) to compute $f(\alpha_t | y_t, \dots, y_1; \theta)$
 3. Use (2) to compute $f(\alpha_{t+1} | y_t, \dots, y_1; \theta)$

¹If α is not stationary, then people generally either choose $f(\alpha_1; \theta)$ to have a large variance, or treat α_1 as an additional parameter to estimate and make $f(\alpha_1; \theta)$ a point mass.

Kalman Filtering

In general, the integrals in (1) and (2) are difficult to compute. However, if we have a linear state-space model with Gaussian errors, then the integrals have simple closed forms. In this case, we write our state-space model as

$$\begin{aligned} y_t &= Z\alpha_t + S\xi_t \\ \alpha_t &= T\alpha_{t-1} + R\eta_t \end{aligned}$$

where $\begin{pmatrix} \eta_t \\ \xi_t \end{pmatrix} \sim N\left(0, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right)$. In this case, all the conditional densities above are Gaussian. Let

$$\begin{aligned} \alpha_t | y_{t-1}, \dots, y_1 &\sim N(\alpha_{t|t-1}, P_{t|t-1}) \\ \alpha_t | y_t, \dots, y_1 &\sim N(\alpha_{t|t}, P_{t|t}) \\ y_t | y_{t-1}, \dots, y_1 &\sim N(y_{t|t-1}, F_t) \end{aligned}$$

Then the above algorithm becomes:

- Let $\alpha_{1|0}$ and $P_{1|0}$ be the marginal mean and variance of α (in case of non-stationarity see the footnote from above).
- For $t = 1, \dots, T$
 1. Use (1) to compute $f(y_t | y_{t-1}, \dots, y_1; \theta)$, i.e.

$$y_{t|t-1} = Z\alpha_{t|t-1} \tag{4}$$

$$F_t = ZP_{t|t-1}Z' + SHS' \tag{5}$$

2. Use (3) to compute $f(\alpha_t | y_t, \dots, y_1; \theta)$ i.e.

$$\alpha_{t|t} = \alpha_{t|t-1} + P_{t|t-1}Z'F_t^{-1}(y_t - y_{t|t-1}) \tag{6}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}Z'F_t^{-1}ZP_{t|t-1} \tag{7}$$

3. Use (2) to compute $f(\alpha_{t+1} | y_t, \dots, y_1; \theta)$ i.e.

$$\alpha_{t+1|t} = T\alpha_{t|t} \tag{8}$$

$$P_{t+1|t} = TP_{t|t}T' + RQR' \tag{9}$$

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