14.454 (Long) Problem Set

Spring 2026

This document collects all long problem set questions for the semester. Each problem header lists the relevant due date. Unless otherwise announced in class, students should submit only the question(s) or subparts designated for submission for that week.

AI guidance. If you think it will be helpful, then you should feel free to explore the use of artificial intelligence (AI) tools such as ChatGPT for the problem sets. Any such use must be appropriately acknowledged and cited. It is each student's responsibility to assess the validity of any AI output that is submitted; you bear the final responsibility. Violations of this policy will be considered academic misconduct.

Schedule

- Problems 1 and 2 due before lecture 5
- Problems 3 and 4 due before lecture 7
- Problems 5 and 6 due before lecture 9
- Problems 7 and 8 due before lecture 11
- Problems 9 and 10 due before lecture 13

[1] If you submit a handwritten solution, you can scan it using an App like Scannable and export it as a PDF file. Please make sure to write in print, not in cursive.

Contents

1 Bernanke-Gertler model

Consider the Bernanke and Gertler (1989) model seen in class. Maintain the same notations and assumptions as in the lecture. For this question, you do not need to derive anything.

1. Explain the financial friction in the Bernanke-Gertler model. Why are internal and external funds not equivalent in the Bernanke-Gertler model? Where does the external finance premium come from?
2. **State verification.** Under this financial friction, lenders audit entrepreneurs reporting k_j with probability $p_j(\ell)$ where ℓ is the size of the loan.
   (a) Describe $p_1(\ell)$ (i.e., the auditing probability when the bad state is reported). How does it depend on ℓ?
   (b) Describe $p_2(\ell)$ (i.e., the auditing probability when the good state is reported). How does it depend on ℓ?
   (c) Explain in words why higher entrepreneurial net worth reduces expected agency costs.
3. **Temporary shock.** Suppose a high θ_t is realized. What are the effects on $E[k_{t+1}]$ and $E[k_{t+2}]$ with and without the financial friction? Explain how the financial friction leads to amplified and persistent economic consequences. Explain how price movements dampen the multiplier. Do the anticipated movements in $E[k_{t+2}]$ affect decisions at t? How?
4. **Permanent shock.** Consider a permanent increase in θ_τ for all $\tau \geq t$.[2] What are the effects on $E[k_{t+1}]$ and $E[k_{t+2}]$ with and without the financial friction? Do the anticipated movements in $E[k_{t+2}]$ affect decisions at t? How?
5. **Asymmetry.** Argue that, under certain conditions, negative shocks may have stronger real effects than positive shocks.
6. **Policy recommendation.** A policymaker is debating whether to make lump-sum transfers away from entrepreneurs to the rest of the population. Assuming that Bernanke-Gertler is the only relevant channel, explain to the policymaker, in one paragraph and without equations, whether this policy would be a good or a bad idea.

2 Kiyotaki-Moore model

Consider the Kiyotaki and Moore (1997) model seen in class. Maintain the same notations and assumptions as in the lecture. Recall that, in equilibrium, the following equation holds:

[2] That is, the expected value of θ_t rises.

$$u(K_t)K_t = (a + q_t)K_{t-1} - Rb_{t-1} \tag{1}$$

where $u(K_t)$ is the user cost of capital. In this exercise, hat variables denote log deviation from the steady-state: $\widehat{x} = \log(x/\overline{x})$ where $\overline{x}$ is the steady-state level of x.

1. Explain the financial friction in the Kiyotaki-Moore model. Why does collateral matter, and why does land command a premium value beyond its productive use?

2. **Interpretation.** Describe each term and interpret equation (1).

3. **Steady state and local dynamics.** Define $\eta^{-1} \equiv \frac{Ku'(K)}{u(K)}$. Using (1), and assuming perfect foresight, find the steady state of the resulting difference equation and log-linearize near that steady state.

4. **Land price equation.** Recall that $u(K_t) = q_t - R^{-1}q_{t+1}$. Find this equation's steady-state and log-linearize this equation around that steady state. Next, iterate forward the log-linearized equation and, using your result in part 3, obtain an equation linking $\widehat{q}_t$ (the price at time t) to $\widehat{K}_t$ (the capital stock at time t).

5. **Temporary productivity shock.** Assume the economy was in steady-state at $t < 0$. At period $t = 0$, an unexpected temporary shock increases productivity to $a(1+\varepsilon)$ with $\varepsilon > 0$.

   (a) Write the "capital equation" at period $t = 0$ and log-linearize it around $\varepsilon = 0$.

   (b) Using your answers above, compute $\widehat{K}_0$ and $\widehat{q}_0$.

   (c) Explain the amplification mechanism of this temporary shock.

   (d) Is this a good model for studying the effects of a temporary decline in productivity (i.e., $\varepsilon < 0$)? Explain.

6. **Static versus dynamic multiplier.** Suppose, hypothetically, that the future land price is pegged at its steady state level, so $q_1 = \overline{q}$.

   (a) Recompute the impact responses of $\widehat{K}_0$ and $\widehat{q}_0$.

   (b) Compare these responses with those in part 5.

   (c) Explain in words the difference between the static multiplier and the dynamic multiplier in this model.

7. **Relative to Bernanke-Gertler.** How do the responses of capital stock (or land) and its price differ from the Bernanke-Gertler model's predictions?

8. **Tighter collateral constraint.** Briefly describe how your answers above would change if the collateral constraint were tightened to $Rb_t \leq \phi q_{t+1}k_t$ for some $\phi < 1$. You don't need to derive anything; intuition is enough.

3 Holmström-Tirole model

Consider the fixed-investment-scale version of Holmström and Tirole (1997) seen in class. Maintain the same notation and assumptions as in lecture. In particular, a project of size I yields return R with probability p_H under diligence and with probability p_L under shirking, where $\Delta p \equiv p_H - p_L > 0$. The entrepreneur obtains private benefit B from the high-private-benefit bad project and $b < B$ from the low-private-benefit bad project. Monitoring by an intermediary rules out the high-private-benefit bad project, but costs the intermediary $c > 0$. Uninformed investors require expected gross return γ, and informed capital earns expected gross return β.

1. Without using equations, explain the two layers of moral hazard in the Holmström-Tirole model. What is the source of the interest rate premium in this model? Why does informed capital earn a higher expected return than uninformed capital?

2. **Direct finance.** Suppose a firm with net worth A raises funds only from uninformed investors.

   (a) Write the entrepreneur's incentive compatibility constraint. What is the maximum pledgeable expected income to uninformed investors?

   (b) Derive the minimum net worth threshold $\bar{A}(\gamma)$ such that firms with $A \geq \bar{A}(\gamma)$ can finance the project directly. Show that $\bar{A}(\gamma)$ is increasing in γ.

3. **Indirect finance.** Now suppose the firm borrows from both an intermediary and uninformed investors, and that the intermediary monitors.

   (a) Write the incentive compatibility constraints for the entrepreneur and the intermediary. What is the maximum pledgeable expected income to uninformed investors?

   (b) If the intermediary must earn gross return β, what is the minimum amount of informed capital invested per monitored firm?

   (c) Derive the minimum net worth threshold $\underline{A}(\gamma, \beta)$ such that firms with $A \geq \underline{A}(\gamma, \beta)$ can obtain financing with monitoring. Show that $\underline{A}(\gamma, \beta)$ is increasing in both γ and β.

4. **Equilibrium.** Using your results above,

   (a) Characterize the equilibrium financing regime of a firm as a function of its net worth A. What are the possible regimes a firm can be in?

   (b) Let $G(A)$ denote the cdf of firm net worth, let aggregate intermediary capital be K_m, and let $S(\gamma)$ (with $S' > 0$) be the aggregate supply of uninformed capital. State the equilibrium conditions that determine β and γ. What are the determinants of the interest rate premium?

5. **Credit crunch.** Consider a reduction in intermediary capital K_m. Briefly explain the effects of this credit crunch event. How do these effects depend on the distribution of net worth?

4 Bernanke-Gertler model with moral hazard

Consider a variant of the Bernanke and Gertler (1989) model, where the friction is moral hazard in a form similar to Holmström-Tirole (1997) (instead of adverse selection). The goal of this exercise is to derive the counterpart of the "capital supply curve" in B-G. The outcome of the indivisible investment project depends on the level of effort by the entrepreneur. In particular, the probability distribution of the outcome of the fixed investment corresponds to:

Actions\Outcome	κ_1	κ_2
Effort: $a = 1$	$1 - \pi^E$	π^E
No effort: $a = 0$	$1 - \pi$	π

where $\kappa_2 > \kappa_1$ and $\pi^E = \Pr(i = 2|a = 1) > \pi = \Pr(i = 2|a = 0)$. The project's outcome $\{\kappa_i\}_{i=1,2}$ is public information (hence contractible).[3] Exerting effort costs the entrepreneur e, while no effort costs the entrepreneur 0. All other features of the model, including the notation, remain the same. Entrepreneurs have all the bargaining power, so all contracts in this exercise maximize the (expected) utility of the entrepreneurs, subject to implementability and limited liability ($c_i \geq 0$ for $i = 1, 2$).

1. **Full information.** Suppose (for now) that effort is verifiable. A contract in this setting specifies consumption for the entrepreneur conditional on effort and the outcome of the project, i.e., $\{c(a, \kappa_i)\}_{a \in \{0,1\}, i}$.

   (a) Argue that the optimal contract induces effort if

   $$\hat{q}_{t+1}(\pi^E - \pi)(\kappa_2 - \kappa_1) > e \tag{2}$$

   where $\hat{q}_{t+1}$ denotes the expected price of capital next period.

   (b) Guess that (2) holds. Show that the implicit equation for capital is given by

   $$\theta f'(k_{t+1})\kappa^E - rx\left(\frac{k_{t+1}}{\eta\kappa^E}\right) - e = 0$$

   where $\kappa^E \equiv (1 - \pi^E)\kappa_1 + \pi^E\kappa_2$ and $\theta = \mathbb{E}\left[\theta_{t+1}\right]$. Let $\bar{\omega}(\hat{q}_{t+1}) > 0$ be such that the entrepreneur invests if $\omega \leq \bar{\omega}(\hat{q}_{t+1})$. When is that guess verified?

[3] In contrast, the original Bernanke-Gertler model assumes that the outcome is private information and the probability distribution is independent of effort.

2. **Incomplete information.** Suppose now that effort is **not** verifiable, i.e. there is moral hazard. A contract in this setting specifies consumption for the entrepreneur conditional on the outcome of the project, i.e., $\{c(\kappa_i)\}_i$. We will now study the optimal contract in a partial equilibrium setting (i.e., taking $\hat{q}_{t+1}$ and S^e as given). In particular, the optimal contract problem is given by $\max\{V_t(0), V_t(1)\}$ where

$$V_t(1) \equiv \max_{\{c_i\}_i} \sum_i c_i \Pr(i|a=1) - e \tag{3}$$

subject to

$$\sum_i c_i \Pr(i|a=1) - e \geq \sum_i c_i \Pr(i|a=0)$$
$$\sum_i (\hat{q}_{t+1}\kappa_i - c_i)\Pr(i|a=1) \geq r(x(\omega) - S^e)$$
$$0 \leq c_1 \text{ and } 0 \leq c_2$$

, and

$$V_t(0) \equiv \max_{\{c_i\}_i} \sum_i c_i \Pr(i|a=0) \tag{4}$$

subject to

$$\sum_i (\hat{q}_{t+1}\kappa_i - c_i)\Pr(i|a=0) \geq r(x(\omega) - S^e)$$
$$0 \leq c_2 \leq c_1.$$

(a) Characterize the optimal contract that induces effort. In particular, show that there exists some $\tilde{\omega}(q_{t+1}, S^e) > 0$ such that this contract (inducing effort) is feasible iff $\omega \leq \tilde{\omega}(\hat{q}_{t+1}, S^e)$. How does $\tilde{\omega}(q_{t+1}, S^e)$ depend on its arguments? Assuming $\tilde{\omega}(\hat{q}_{t+1}, S^e) \leq \bar{\omega}(\hat{q}_{t+1})$, show that the entrepreneur would invest in the project (with the optimal contract that induces effort) iff $\omega \leq \tilde{\omega}(\hat{q}_{t+1}, S^e)$.

(b) Characterize the optimal contract that induces no effort. In particular, show that there exists some $\hat{\omega}(q_{t+1}) > 0$ such that the entrepreneur would invest in the project (with the optimal contract that induces no effort) iff $\omega \leq \hat{\omega}(q_{t+1})$. Explain why $\hat{\omega}(q_{t+1})$ does not depend on S^e.

(c) Suppose that (2) holds, so that inducing effort is socially efficient. Show that $\hat{\omega}(q_{t+1}) < \bar{\omega}(\hat{q}_{t+1})$. Find the optimal investment decisions under the assumption that $\hat{\omega}(q_{t+1}) < \tilde{\omega}(\hat{q}_{t+1}, S^e)$ (Keep this assumption below).

(d) Derive the "capital supply curve" and compare it to the one with full information (item 1). Describe the dynamic impact of a high-θ_t realization with and without the financial friction. How do entrepreneurs' savings affect the "capital supply curve" in this setting (hidden action) and in the Bernanke and Gertler setting (hidden outcomes)?

5 Bank runs and contagion

There are $2N$ regions in a country. Each region $i = 1, \ldots, 2N$ is inhabited by a unit mass of consumers that live for three periods $t \in \{0, 1, 2\}$ and are all ex-ante identical. At the beginning of $t = 1$, a fraction π_i of consumers suddenly become impatient and only care about consumption at $t = 1$. The remaining fraction $1 - \pi_i$ is patient and only values consumption at $t = 2$. The ex-ante expected utility of a consumer in town i is:

$$u(c_1, c_2) = \pi_i \ln c_1 + (1 - \pi_i)\beta \ln c_2.$$

At period 0, all consumers are endowed with one unit of consumption good and have two potential technologies. First, a storage technology with a return 1 is available between any two periods. Second, they can invest in a long-run project which yields R at $t = 2$, but if liquidated at $t = 1$, yields $L < 1$. Throughout this question, suppose $\beta R > 1$.

5.1 Diamond-Dybvig model (1983)

First, suppose all π_i are equal to π, so all regions are identical.

1. **Equilibrium without financial markets.** Characterize the investment decisions and equilibrium consumption of early and late consumers in autarky (no financial markets).

2. **Equilibrium with bond markets.** Suppose now that at $t = 1$, agents can trade a bond that costs p at $t = 1$ and yields a unit of output at $t = 2$. Characterize the competitive equilibrium. How does it compare with autarky?

3. **Social planner's allocation.** Suppose there is a social planner. Characterize the constrained-efficient allocation (i.e. the planner cannot observe the type of agents in period 1). Show that the IC constraint does not bind.

4. **Equilibrium with banks.** Suppose now that we implement the allocation in (3) with a bank and deposit contracts that give the consumer the right to withdraw (c_1^*, c_2^*). The bank can also invest in the same long-run project. If the bank does not have enough resources in $t = 1$, it prioritizes honoring its period 1 commitments by liquidating some of the long-term asset; in each period, the amount the bank distributes is equally split among those who withdraw. The individuals have two choices: deposit 1 in $t = 0$ or remain in autarky forever (no side trades allowed). Let λ be the proportion of individuals who withdraw in period 1.

   (a) Show that all impatient individuals withdraw at $t = 1$.

(b) Find the best response of a patient consumer at $t = 1$, conditional on the proportion being λ.

(c) Find all the Nash equilibria at $t = 1$. How does the set of equilibria depend on β?

5. **Policy intervention.** Suppose one of the following policies is introduced after deposits are made: (i) credible deposit insurance financed by taxes at $t = 2$; (ii) suspension of convertibility once withdrawals exceed a threshold $\bar{\lambda}$; and (iii) lender-of-last-resort lending against the long asset at the actuarially fair rate. For each policy,

   (a) Can the efficient allocation still be implemented? Does the run equilibrium survive?

   (b) Explain what fiscal or collateral condition is needed for the policy to be credible.

5.2 Allen-Gale model (2000)

Now impose some differences between the regions in the following way: assume π_i takes one of the two values $\omega_H > \omega_L$, and the realization depends on a state $S \in \{S_1, S_2\}$ with equal probability in the following way:

- When $S = S_1$, $\pi_i = \omega_H$ if and only if i is odd,
- When $S = S_2$, $\pi_i = \omega_H$ if and only if i is even.

6. **Social planner's allocation.** Suppose there is a social planner over all $2N$ regions. Characterize the constrained-efficient allocation, and show that it achieves the same outcome as in parts 1-3 with $\pi = (\omega_H + \omega_L)/2$. Call this value γ.

7. **Equilibrium without interbank market.** Suppose that now we want to decentralize the allocation in (6). Assume that each region has a continuum of identical banks; the representative bank in i accepts deposits and gives the consumer the right to withdraw (c_{1i}, c_{2i}). Explain briefly why the stochasticity of π_i forbids banks in each region from independently providing the same allocation as in part (6). What does it mean for there to be "no overall shortage of liquidity, it is just badly distributed" (Allen and Gale)?

8. **Equilibrium with interbank market.** Suppose that banks are allowed to exchange deposits in $t = 1$.

   (a) (*Complete markets*) Suppose every bank is connected to all others in the following way: each bank in region i holds deposits $z^i = (\omega_H - \gamma)/N$ in all other banks. Show that this is feasible (does not violate any budget constraints) and achieves the first-best without having to liquidate additional assets in $t = 1$.

(b) (*Incomplete markets*) Suppose each bank in region i holds deposits $z^i = (\omega_H - \gamma)$ in banks in region $i+1$ (and bank in region $2N$ holds deposits in region 1.) Show that this also achieves the first-best.

9. **Contagion.** Now suppose that in period 1, an unexpected event happens - state $S = \bar{S}$ is realized, where $\pi_i = \gamma$ for all $i \geq 2$, but $\pi_1 = \gamma + \epsilon$: the average demand for liquidity across all regions is identical, except it's slightly higher in region 1. Maintain the same assumptions and notations as in the lectures (regarding pecking order and liquidation.)

   (a) Define what the "buffer" of a bank with early consumers ω is, and write down its formula $b(\omega)$.

   (b) (*Complete markets*) Suppose we're in the complete market structure in 8-(a). Argue that for any shock ϵ (with $\gamma + \epsilon < 1$), as long as N is large enough, the shock $\bar{S}$ will be contained in one region.

   (c) (*Incomplete markets*) Suppose we're in the incomplete market structure in 8-(b). Write down the two inequalities which, when satisfied, will trigger the entire system to break down. Explain in words what each inequality means, and how it will break the system. (No need to prove this rigorously.)

6 Diamond-Dybvig in a small open economy

Consider a small open economy with three dates, $t \in \{0, 1, 2\}$, and a continuum of ex ante identical domestic agents of mass 1. Each agent is endowed with $e > 0$ units of consumption good at $t = 0$. There is one long term domestic technology: one unit invested at $t = 0$ yields $R > 1$ units at $t = 2$, but only $L < 1$ units if liquidated early at $t = 1$. There is also access to a world capital market: one unit stored in the world market yields one unit either at $t = 1$ or $t = 2$. Domestic agents can borrow from abroad up to an exogenous ceiling $f > 0$. At $t = 1$, each domestic agent privately learns her type: with probability π she is impatient and values only $t = 1$ consumption; with probability $1 - \pi$ she is patient and values only $t = 2$ consumption. Preferences are

$$\pi \ln(c_1) + (1 - \pi) \ln(c_2). \tag{5}$$

This is the same preference and timing structure as the Diamond and Dybvig (1983) model, but now the bank can borrow abroad. Let k denote investment in the long asset at $t = 0$, d net foreign borrowing at $t = 0$, b additional foreign borrowing at $t = 1$, I liquidation of the long asset at $t = 1$, and (c_1, c_2) the consumption to impatient and patient agents.

1. **The constrained efficient allocation.** The planner's problem is to choose $\{c_1, c_2, k, d, b, I\}$

to maximize (5) subject to

$$k \leq e + d \tag{6}$$
$$\pi c_1 \leq b + LI \tag{7}$$
$$(1-\pi)c_2 + d + b \leq R(k - I) \tag{8}$$
$$d \leq f \tag{9}$$
$$d + b \leq f \tag{10}$$
$$c_1 \leq c_2 \tag{11}$$

and nonnegativity constraints (i.e., $c_1, c_2, k, I \geq 0$). Let us denote the solution for this problem as $\{c_1^*, c_2^*, k^*, d^*, b^*, I^*\}$. Throughout this problem, suppose that f is "sufficiently large."[4]

(a) Explain each of the constraints in this optimization problem.

(b) Argue that $I^* = 0$ and constraints (7) and (10). Show that the constraints can be simplified to a single implementability constraint

$$R\pi c_1 + (1-\pi)c_2 = Rw \quad \text{where} \quad w \equiv e + \frac{(R-1)}{R} f.$$

(c) Find $\{c_1^*, c_2^*, k^*, d^*, b^*\}$.

2. **Equilibrium with demand deposits.** Suppose the bank offers a demand deposit contract that gives each depositor the right to withdraw c_1^* at $t = 1$ and c_2^* at $t = 2$. Withdrawals at $t = 1$ are served sequentially. Assume for now that the bank always repays foreign creditors.

   (a) Show that if the bank must always preserve enough resources to repay total foreign debt f, then the maximum liquidation consistent with that promise is $I^+ = k^* - f/R$.

   (b) If all depositors attempt to withdraw at $t = 1$, show that the bank fails if $z^+ \equiv c_1^* - (b^* + LI^+) > 0$. Using the solution of the planner's optimization problem, is a bank run an equilibrium for this economy?

   (c) What is the interpretation of z^+? Why is this the natural open economy analogue to Diamong-Dybvig illiquidity?

3. **Foreign creditors and ongoing lending.** Relax the assumption that foreign creditors always lend b in period 1 during a run. Suppose instead that, if a run occurs, no new foreign lending is extended.

   (a) Show that if the bank still repays its initial foreign debt d^*, then the maximum liquidation is $I^a = k^* - d^*/R$.

[4] In particular, suppose $f > \frac{\pi}{1-\pi}\frac{R}{R-1}e$.

(b) Show that in this case the bank fails during a run if $z^a \equiv c_1^* - LI^a > 0$.

(c) Show that $z^a > z^+$. Relative to the bank run in described part 2, what does this condition imply for the vulnerability of the bank?

(d) Suppose that the model's parameterization were different and we get $z^a > 0 > z^+$. What would this condition imply for the origin of a bank run?

(e) Consider a case where the bank borrows the full amount f at $t = 0$, to be repaid at $t = 2$, and invests part of it in the long asset and holds b^* as liquid reserves. Would the bank be vulnerable to foreign creditors' confidence?

4. **Short term external debt.** Suppose the bank's initial foreign borrowing d^* is short term, so it comes due in period 1 and must be rolled over.

(a) If a run occurs and foreign creditors refuse to roll over the debt, show that the bank fails if $z^b \equiv c_1^* + d^* - Lk^* > 0$.

(b) Show that, whenever $d^* > 0$, $z^b > z^a > z^+$. Explain why short term external debt makes the financial system strictly more fragile.

7 RE Bubbles with incomplete markets

Consider an OLG economy, where households live for two consecutive periods. There is no uncertainty, and no population growth. There is a single final good (in every period), which can be either consumed or invested in capital. Households' preferences (generation t) are represented by $U(c_t^y, c_t^o) = \beta \ln(c_t^y) + (1-\beta)\ln(c_t^o)$. Households supply labor inelastically when young ($L = 1$ per capita). There is a representative firm, with Cobb-Douglas production function $Y_t = K_t^\alpha (A_t L)^{1-\alpha}$. Productivity A_t grows at rate g. Capital must be invested one period in advance and does not depreciate. Markets are competitive.

1. **Benchmark: No bubbles.** Suppose (for now), that capital is the only asset available for each generation t.

   (a) Let $k_t = K_t/A_t$ and $w_t \equiv W_t/A_t$. Derive the demand and supply of capital k (i.e., savings) as a function of current wages (w). Show that in equilibrium this implies a relationship between the rental rate of capital r_{t+1} and wages w_t.

   (b) Using the first-order conditions of the firms' problem, derive the following first-order difference equation in r_t (i.e., r_{t+1} as a function of r_t).

   (c) Characterize the steady state(s) of this economy. When is the economy dynamically inefficient?

2. **Adding bubbles.** Suppose in the following that there is an irreproducible and useless asset ("bubble" or "money"), B_t, traded at each period $t \geq 0$.

   (a) Let $b_t \equiv B_t/A_t$ denote the size of the bubble per efficiency unit. Show that now equilibrium can be described by the following two equations,

   $$(1+g)k_{t+1} + b_t = (1-\beta)(1-\alpha)k_t^\alpha \quad \text{and} \quad b_{t+1} = \frac{1+r_{t+1}}{1+g} b_t.$$

   (b) Let $\bar{r}$ denote the steady state of the economy without bubbles. Show that if the steady state is dynamically efficient, a bubble cannot exist in this economy.

   (c) Assume some parameter values such that the steady state is dynamically inefficient and draw a phase diagram.

   (d) Show that, given k_0, there is a unique initial bubble b_0 that is consistent with an asymptotic interest rate of g. How does b_0 change with k_0?

   (e) Pick two feasible initial values $b_0 > b_0'$ and compare their equilibrium trajectories. Discuss the welfare properties of equilibria with and without bubbles. In what sense do bubbles "crowd out" investment in this economy?

3. **Endogenous growth.** Allow now for endogenous growth: $A_t = \mu K_t$ for some $\mu > 0$. Let $1 + g_t \equiv A_t/A_{t-1}$.

(a) Show that the equilibrium is entirely characterized by the following difference equation:

$$b_{t+1} = \frac{1}{\mu} \frac{1 + \alpha \left(\frac{1}{\mu}\right)^{\alpha-1}}{(1-\beta)(1-\alpha)\left(\frac{1}{\mu}\right)^{\alpha} - b_t} b_t$$

for some arbitrary b_0. Solve for the steady states associated to this difference equation and discuss stability. Show that there exists some $b^* > 0$ such that $b_t \leq b^*$ for all t, for any equilibrium.

(b) Pick two feasible initial values $b_0 > b'_0$ and compare the equilibrium sequences for output growth associated with those. Are the bubbles welfare-improving when the economy is dynamically inefficient?

8 Credit booms (Due date: 05/01)

This exercise goes through a simpler version of Lorenzoni (2008), where there is no uncertainty and no financial contracting. There are three periods $t = 0, 1, 2$, two groups of agents (households and entrepreneurs), and two goods: consumption goods ("goods") and capital. Preferences are given by $u = c_0 + c_1 + c_2$ for both groups of agents. Entrepreneurs are endowed with n goods at $t = 0$; households are endowed with e_0 and e_1 goods in periods $t = 0, 1$ respectively.

Goods are perishable, but entrepreneurs can convert goods into capital one-for-one at $t = 0$. Capital requires a per-unit maintenance cost of z goods at $t = 1$ in order to produce goods at $t = 2$: if the maintenance costs are not paid, that unit of capital becomes useless. At $t = 2$, a maintained unit of capital will produce A goods if it's operated by entrepreneurs; instead, if households operate k^H units of capital, they will produce $F(k^H)$ goods, where F is a differentiable concave function.

Assume that no intertemporal contracts are enforceable, so agents cannot borrow or lend; likewise, capital will be managed by its owner, so it's impossible for entrepreneurs to manage capital on behalf of households. There is, however, a competitive market for maintained capital at $t = 1$ with a price q.

1. **Resource constraints.** In this economy, the aggregate resource constraints are given by

the following four equations:

$$c_0^E + c_0^H + k \leq n + e_0 \tag{12}$$
$$c_1^E + c_1^H + z\alpha k \leq e_1 \tag{13}$$
$$k^E + k^H \leq \alpha k \tag{14}$$
$$c_2^E + c_2^H \leq Ak^E + F(k^H) \tag{15}$$

What does α represent here?

2. **Entrepreneur.** The entrepreneurs take price q as given and solve the following problem:

$$\max_{c_0^E, c_1^E, c_2^E, k, s, k^E, \alpha} c_0^E + c_1^E + c_2^E \tag{16}$$

such that

$$c_0^E + k \leq n \tag{17}$$
$$c_1^E + \alpha z k \leq sq \tag{18}$$
$$k^E \leq \alpha k - s \tag{19}$$
$$c_2^E \leq Ak^E \tag{20}$$

and $c_t^E, k, k^E \geq 0$, $0 \leq \alpha \leq 1$, and $0 \leq s \leq \alpha k$. What does s represent?

3. **Household & competitive equilibrium.** Set up the household's maximization problem, taking q as given and choosing c_0^H, c_1^H, c_2^H, k^H. Define a competitive equilibrium in this economy.

4. **Characterization.** Suppose conditions such that all solutions are interior.[5]

   (a) Set up the problem for the entrepreneur in period 1 who holds k units of capital. Denote their period 1 onwards utility by $V(k, q)$.

   (b) Simplify the entrepreneur's problem at $t = 0$ (part 2), and solve for k as a function of q, n.

   (c) Turn to the household's problem. Combine the first-order condition with the market clearing condition $s = k^H$ to find an implicit equation that would pin down the relationship between q and k.

   (d) Explain why the price $q(k)$ is decreasing in k. (You can explain in words, no math required)

5. **Planner's problem.** The constrained optimization problem for the social planner at $t = 0$ is as follows: the planner maximizes entrepreneur's utility subject to delivering a minimum utility Π to households. The planner is free to dictate investment and make transfers at $t = 0$, but cannot intervene in $t = 1, 2$.[6]

[5] Specifically, those conditions are: $A > 1 + z$, $F'(0) \in (\frac{Az}{A-1}, A)$, $-F''(x)x/F'(x) < 1$, $F'(\frac{A-1}{A}n) < \frac{Az}{A-1}$,$e_1 > zn$.

[6] This is equivalent to a Ramsey policy problem with a tax on investment and lump-sum redistribution - see Davila and Korinek (2017).

(a) Set up the constrained optimization problem, writing out each constraint. The planner should seek to maximize $c_0^E + V\left(k, q(k)\right)$.

(b) Show that the planner's objective, as a function of k, is

$$W(k) = n + e_0 + e_1 - zk + F\left(s(q(k), k)\right) - \Pi - k + V\left(k, q(k)\right)$$

(c) Taking $q(k)$ decreasing in k as given, show that the constrained optimum k^* is lower than the equilibrium investment k: that is, the social planner can obtain a Pareto improvement by lowering investment relative to the competitive equilibrium.

(d) Explain in words what the pecuniary externality is in this model, and its relation to the inefficiency of the competitive equilibrium here.

9 Liquidity Trap

This problem is based on Korinek-Simsek (2016). There are two types of households $h = \{b, l\}$, each with unit mass. They maximize their utility

$$\sum_{t=0}^{\infty} (\beta^h)^t \log\left(c_t^h - \frac{(n_t^h)^{1+\psi}}{1+\psi}\right)$$

where c^h is consumption, n^h is labor, and $\beta^h$is the discount factor for household of type h. Assume $\psi > 0$ and $\beta^b < \beta^l \leq 1$. Agents face a borrowing constraint $d_t^h \leq \phi_t$ where d_t^h is the amount of consumption units owed at t. Competitive firms, owned by households, produce the consumption good using a linear technology $y_t = n_t$. Let $\phi_t = \bar{\phi}$ for $t \leq 1$ and $\phi_t = \underline{\phi}$ for $t \geq 2$, with $\underline{\phi} < \bar{\phi}$. At $t = 0$, $d_0^b = -d_0^l = d_0$ is given.

1. **Flexible price equilibrium.** Suppose prices are flexible. What is the competitive equilibrium in this economy?
2. **Liquidity trap.** Show that for d_0 large enough or β^b small enough, $r_2 < 0$: the real interest rate at $t = 2$ is negative.
3. **ZLB and Macroprudential policies.** Assume that goods prices are fully rigid, so $\bar{p} = 1$ and firms have pre-committed to satisfy any demand at this price.
   (a) The Central Bank sets the nominal interest rate such that the real interest rate corresponds to the flexible-price economy, but is subject to the Zero Lower Bound – if the resulting nominal interest rate is negative, the nominal rate is set at 0. Characterize the new competitive equilibrium.
   (b) Suppose the Central Bank is considering setting a debt ceiling at $t = 1$, i.e. $d_1 \leq \bar{d}$. How does this affect the welfare of each group?
   (c) Suppose the Central Bank promises a boom from $t = 2$ onwards, i.e. $y_t - y^f > 0$ for all $t \geq 2$ where $y^f$is the output level in the flexible-price economy. How does this affect the welfare of each group?
4. Discuss the channel through which these policies affect welfare. What is the externality associated with debt? What is the underlying friction? In what sense is interest rate policy inferior to macroprudential policies in dealing with excessive leverage?

10 Bubbles & capital flow volatility

In this problem, we will derive some of the conclusions of Caballero-Krishnamurthy (2006). Maintain the same notations and assumptions as in the lecture. Recall that the following equation holds:

$$\frac{1}{2}p_{t+1}I_{t+1} \leq \frac{1}{2}\psi R K_{t+1}.$$

1. Interpret this equation. What does ψ represent? Explain why this equation rules out using domestic assets or output as collateral (in other words, how would this equation change if entrepreneurs can use their own output as collateral?)

2. **Equilibrium.** Show that the equilibrium price p_{t+1} must satisfy $1 \leq p_{t+1} \leq R$. Furthermore, assume $\psi R < 1$ and $K_t = W_t$ and show that $p_{t+1} = 1$.

3. **Dynamic inefficiency.** Assume that $W_{t+1} = (1+g)W_t$ where $g > r^*$:

   (a) Show that this economy is lending an exponential amount of international goods abroad each period.

   (b) Show that this economy is dynamically inefficient: that is, show that a social planner can redistribute the endowments in a way that makes everyone better off (Samuelson 1958).

4. **Bubbles.** Assume the existence of a bubble ("real estate") B_t that holds no value (Tirole 1985). Suppose that the bubble grows at a rate g if the bubble does not burst (i.e., next generation youth invests in the bubble), but it crashes to a value of 0 if the young decide to save their international goods abroad. Suppose that in each period, the bubble crashes with probability λ.

   (a) What is the expected return $\hat{r}^b$ from investing in the bubble?

   (b) If λ is small and $g - r^*$ is large enough, show that young agents will invest at least some of their goods in the bubble.

   (c) Let $\bar{r}_{t+1} \in \{g, -1\}$ be the realized return on the bubble at period $t+1$, and $\tilde{p}_{t+1}$ be the corresponding price of international goods in each state. If α_t is the proportion of wealth young agents invest in the bubble. Determine the consumption of entrepreneurs and bankers in period $t+1$ as a function of α_t .

   (d) Set up the young agent's optimization problem in period t. What equation should hold for an interior solution?

   (e) Set up the market clearing conditions in each state (crash/no-crash) and solve for $\tilde{p}_{t+1}$ in each state. Discuss why the economy faces a risk/return trade-off associated with the bubble (when $\lambda > 0$).

5. **Excess volatility.** In the previous part, we solved for the equilibrium α_t. We're now interested in the welfare-maximizing α_t, α_t^*.

   (a) Set up the social planner's problem for each generation t such that the bubble has not crashed.

(b) Argue that if $p^C_{t+1} < R$, where p^C_{t+1} is the price $\tilde{p}_{t+1}$ after crash in period $t+1$ (that you identified in 4-e), then the market equilibrium α_t is higher than the welfare-maximizing α^*_t. Why is this happening?

(c) What is the role of the "credit crunch" here?

6. **Capital inflow sterilization.** Implementing the efficient allocation (math is not required).

   (a) Explain how the government issuing bonds and taxing the endowment of young agents can achieve the welfare-maximizing outcome. How does this resolve the credit crunch? Why does this policy require strong fiscal capabilities?

   (b) Does capital flows sterilization solve the dynamic inefficiency problem in this economy? Discuss (in words) how public debt can reduce fragility and solve the dynamic inefficiency problem.