14.454 (Long) Problem Set
Solutions

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1 Holmtröm-Tirole (1997)

1. There are two moral hazard problems. First, in the absence of proper incentives or outside monitoring, a firm may deliberately reduce the probability of success in order to enjoy a private benefit. To get external financing, total surplus must be redistributed. Due to limited liability, the only way a firm transfers some of the surplus back to investors is by investing its own capital. Capital-poor firms will be unable to invest, because they do not have the means to redistribute surplus.

   Second, monitoring is privately costly. The intermediary has to pay a nonverifiable amount to reduce the firm's private benefit from shirking. While each intermediary has the physical capacity to monitor an arbitrary number of firms, this moral hazard problem puts a limit on the actual amount of monitoring that will take place: it forces intermediaries to inject some of their own capital into firms that they monitor, making the aggregate amount of monitoring capital one of the important constraints on aggregate investment.

   The interest premium comes from the scarcity and costliness of informed capital. Uninformed investors just supply funds and require expected return γ. Informed capital must both cover the monitoring cost c and satisfy the intermediary's own incentive constraint, so it earns a higher expected return β. Thus β exceeds γ because monitoring is costly and intermediaries must be incentivized.

2. **Direct finance.** With only uninformed investors, if the project succeeds the entrepreneur gets R_f and investors get R_u, with $R = R_f + R_u$.

   (a) The entrepreneur must prefer diligence to the high-private-benefit bad project:

   $$p_H R_f \geq p_L R_f + B \Longrightarrow R_f \geq \frac{B}{\Delta p}.$$

*The solutions for Q4 are based on those written by Sebastián Fanelli and Nathan Zorzi.

Hence the maximum payoff left for uninformed investors in the success state is $R_u = R - \frac{B}{\Delta p}$ and the maximum pledgeable expected income is

$$p_H \left[R - \frac{B}{\Delta p} \right].$$

(b) If the firm has net worth A, uninformed investors must supply $I - A$. Their participation requires

$$\gamma(I - A) \leq p_H \left[R - \frac{B}{\Delta p} \right].$$

Therefore, the minimum net worth needed for direct finance is

$$\overline{A}(\gamma) = I - \frac{p_H}{\gamma} \left[R - \frac{B}{\Delta p} \right].$$

Function $\overline{A}(\gamma)$ is crearly increasing in γ.

3. **Indirect finance.**

(a) If the intermediary monitors, the entrepreneur can still choose between the good project and the low-private-benefit bad project. So the entrepreneur's IC is

$$p_H R_f \geq p_L R_f + b \Longrightarrow R_f \geq \frac{b}{\Delta p}.$$

The intermediary must also want to monitor:

$$p_H R_m - c \geq p_L R_m \Longrightarrow R_m \geq \frac{c}{\Delta p}.$$

Therefore, the maximum success-state payoff left for uninformed investors is $R_u = R - \frac{b+c}{\Delta p}$ and the maximum pledgeable expected income is

$$p_H \left[R - \frac{b+c}{\Delta p} \right].$$

(b) If the intermediary must earn gross expected return β on its informed investment I_m, then $\beta I_m = p_H R_m$. Using the minimum IC value $R_m = \frac{c}{\Delta p}$,

$$I_m(\beta) = \frac{p_H c}{\beta \Delta p}.$$

This is the minimum informed capital needed per monitored firm.

(c) With monitoring, uninformed investors provide $I - A - I_m(\beta)$. Their participation requires

$$\gamma(I - A - I_m(\beta)) \leq p_H \left[R - \frac{b+c}{\Delta p} \right].$$

Hence the minimum entrepreneurial net worth needed for monitored finance is

$$\underline{A}(\gamma, \beta) = I - \frac{p_H}{\beta} \frac{c}{\Delta p} - \frac{p_H}{\gamma} \left[R - \frac{b+c}{\Delta p} \right].$$

Clearly, this function is increasing in both β and γ.

4. **Equilibrium**

   (a) In the model, a firm's net worth determines its debt capacity. Firms that take on too much debt in relation to equity will not have a sufficient stake in the financial outcome and will therefore not behave diligently. Assuming that investment projects are of fixed size, only firms with sufficiently high net worth will be able to finance investments directly, i.e., jointly with uninformed investors. Firms with low net worth have to turn to financial intermediaries (informed investors), who can reduce the demand for collateral by monitoring more intensively. However, all firms cannot be monitored in equilibrium, because intermediaries, like firms, must invest some of their own capital in a project in order to be credible monitors. In the market for monitoring, the equilibrium interest premium paid on monitoring capital is then determined by the relative amounts of aggregate firm and aggregate intermediary capital.

      i. If $A < \underline{A}(\gamma, \beta)$, the firm cannot raise enough funds even with monitoring, so it is not financed.

      ii. If $\underline{A}(\gamma, \beta) \leq A < \overline{A}(\gamma)$, the firm is financed only with the help of an intermediary, so it uses indirect finance.

      iii. If $\overline{A}(\gamma) \leq A$, the firm can finance directly with uninformed investors, so it uses direct finance.

   (b) The market for informed capital clears when aggregate intermediary capital equals demand from all monitored firms:

$$K_m = \left[G(\overline{A}(\gamma)) - G(\underline{A}(\gamma, \beta))\right] I_m(\beta).$$

   The market for uninformed capital clears when supply equals total demand from both directly financed and monitored firms:

$$S(\gamma) = \int_{\underline{A}(\gamma,\beta)}^{\overline{A}(\gamma)} [I - A - I_m(\beta)] dG(A) + \int_{\overline{A}(\gamma)}^{\infty} [I - A] dG(A)$$

   The interest rate premium between informed and uninformed capital is therefore determined by the relative scarcity of intermediary capital, the supply of uninformed funds, and the distribution $G(A)$. Note that firms with $A > I$ will act as uninformed investors too.

5. A reduction in K_m is a credit crunch. It raises the scarcity of informed capital, so monitored finance becomes harder to obtain. The threshold $\underline{A}(\gamma, \beta)$ rises, aggregate investment falls, and the firms hit first are the weakest firms that previously depended on monitoring. This is the model's flight-to-quality result.

   Under proper assumptions on the distribution of net worth $G(A)$, when monitoring capital decreases, the market debt interest rate falls and the interest rate spread increases. For having this effect, it must be the case that the mass of firms at the lower net worth bound for indirect financing is large enough.

2 Bernanke-Gertler (1989) with Moral Hazard

1. Suppose (for now) that effort is verifiable. A contract in this setting specifies consumption for the entrepreneur conditional on effort and the outcome of the project, i.e., $\{c(a,\kappa_i)\}_{a\in\{0,1\},i}$. Then, the optimal contract maximizes the (expected) utility of the entrepreneurs subject to the lender's participation constraint and limited liability. We can ignore IC constraints as the desirable action can be enforced: suppose the optimal effort is $a^* \in \{0,1\}$, then setting $c(-a^*,\kappa_i) = 0$ provides the proper incentives (as long as (2) holds). Given $\hat{q}_{t+1}$, the optimal contract problem for the $\omega-$entrepreneur is

$$\max_{a,c(\cdot)} \sum_i c(a,\kappa_i)\text{Pr}(i|a) - \text{Cost}(a) \tag{1}$$

subject to $c(a,\kappa_i) \geq 0$ for all a, i; and

$$\sum_i \left(\hat{q}_{t+1}\kappa_i - c(a,\kappa_i)\right)\text{Pr}(i|a) \geq r\left(x(\omega) - S^e\right).$$

Hence, the participation constraint binds at the optimal contract.[1]

(a) The (expected) payoff for the entrepreneur when he optimally implements effort $a \in \{0,1\}$ is

$$V_t^{fb}(a) \equiv \sum_i c(a,\kappa_i)\text{Pr}(i|a) - \mathbf{1}_{\{a=1\}}e = \hat{q}_{t+1}\sum_i \kappa_i\text{Pr}(i|a) - r\left(x\left(\omega\right) - S^e\right) - \mathbf{1}_{\{a=1\}}e$$

Thus,

$$V_t^{fb}(1) > V_t^{fb}(0) \iff \hat{q}_{t+1}(\pi^E - \pi)(\kappa_2 - \kappa_1) > e. \tag{2}$$

Therefore, $a^* = 1$ and the limited liability constraints are satisfied.

(b) Guess that (2) holds at equilibrium, so that $a^* = 1$. The entrepreneur has two options: (i) investing, with the optimal contract, with (expected) payoff given by (1); and (ii) the storage technology, with payoff rS^e. Therefore, the entrepreneurs invest iff $\omega \leq \bar{\omega}(\hat{q}_{t+1})$, with

$$\hat{q}_{t+1}\kappa^E - r\left(x\left(\bar{\omega}(\hat{q}_{t+1})\right) - S^e\right) - e = rS^e \iff \hat{q}_{t+1}\kappa^E - rx\left(\bar{\omega}(\hat{q}_{t+1})\right) - e = 0 \tag{3}$$

where $\kappa^E \equiv (1-\pi^E)\kappa_1 + \pi^E\kappa_2$. Note that $\bar{\omega}(\hat{q}_{t+1})$ increases with $\hat{q}_{t+1}$, since the return on the investment project increases, and does not depend on S^e, since the payoff associated to both options increases linearly with S^e, with slope r. On the one hand, the higher S^e, the lower the borrowing and thus the higher the payoff associated to the contract. On the other hand, the higher S^e, the higher the savings and the higher the return.

Thus, capital supply satisfies:

$$k_{t+1} = \eta\bar{\omega}(\hat{q}_{t+1})\kappa^E. \tag{4}$$

[1] You can assume $\hat{q}_{t+1}\sum_i \kappa_i\text{Pr}(i|a) \geq r\left(x(\omega) - S^e\right)$ for each $a \in \{0,1\}$.

Capital demand satisfies:

$$\hat{q}_{t+1} = \theta f'(k_{t+1}). \tag{5}$$

From (3)-(5),

$$\theta f'(k_{t+1})\kappa^E - rx\left(\frac{k_{t+1}}{\eta\kappa^E}\right) - e = 0. \tag{6}$$

Thus, the guess is verified iff

$$\frac{1}{\kappa^E}\left[rx\left(\frac{k_{t+1}}{\eta\kappa^E}\right) + e\right](\pi^E - \pi)(\kappa_2 - \kappa_1) > e$$

where $\{k_{t+1}\}_{t\geq 0}$ solves (6).

2. Suppose now that effort is not verifiable, i.e. there is moral hazard. A contract in this setting specifies consumption for the entrepreneur conditional on the outcome of the project, i.e., $\{c(\kappa_i)\}_i$.

   (a) By definition, a contract inducing effort is feasible iff there exists some (c_1, c_2) that satisfies (IC)-(LL). Note that if (c_1, c_2) is feasible, then (c_1', c_2') is feasible, with $c_1' = 0$ and

   $$c_2' = \frac{1}{\pi^E}\left((1-\pi^E)c_1 + \pi^E c_2\right);$$

   since (PC) holds, $c_2' \geq c_2 \geq 0$ so that (LL) holds, and

   $$\sum_i c_i'\Pr(i|a=1) - \sum_i c_i'\Pr(i|a=0) = \left(\frac{\pi^E - \pi}{\pi^E}\right)\left[(1-\pi^E)c_1 + \pi^E c_2\right] \geq 0$$

   since $\pi^E > \pi$ and $c_i \geq 0$, so that (IC) holds. Hence, $c_1 = 0$ is w.l.o.g. for feasibility. Similarly, if (c_1, c_2) is feasible, then (c_1', c_2') is feasible with $c_1' = c_1$ and c_2' such that

   $$\hat{q}_{t+1}\kappa^E - \sum_i c_i'\Pr(i|a=1) = r(x(\omega) - S^e) \tag{7}$$

   since (PC) holds, $c_2' \geq c_2 \geq 0$ so that (LL) holds, and (IC) holds. Then, binding (PC) is w.l.o.g. for feasibility. Therefore, a contract inducing effort is feasible iff there exists (c_1, c_2) with $c_1 = 0$ satisfying (IC) and

   $$\pi^E c_2 = \hat{q}_{t+1}\kappa^E - r(x(\omega) - S^e). \tag{8}$$

   The contract inducing effort is feasible iff $\omega \leq \tilde{\omega}(\hat{q}_{t+1}, S^e)$ where

   $$\hat{q}_{t+1}\kappa^E - r\left(x\left(\tilde{\omega}\left(\hat{q}_{t+1}, S^e\right)\right) - S^e\right) \equiv \frac{\pi^E}{\pi^E - \pi}e > 0. \tag{9}$$

   Note that (LL) is also satisfied whenever $\omega \leq \tilde{\omega}(\hat{q}_{t+1}, S^e)$. Moreover, this contract is the same as the one characterized in Part 1.

   Function $\tilde{\omega}(\hat{q}_{t+1}, S^e)$ is increasing in both arguments. The higher the (expected) price of capital and the higher net worth (and thus the lower borrowing), the higher the (expected) payoff from the contract, from (8). Therefore, the lower the incentives to choose low effort over high effort, for any ω, so that the threshold is increasing.

An entrepreneur with ω has two options: (i) invest in the project (with the optimal contract that induces effort), with expected payoff

$$\pi^E c_2 - e = \hat{q}_{t+1}\kappa^E - r(x(\omega) - S^e) - e$$

or (ii) invest in savings technology, with return: rS^e. The entrepreneur chooses the project iff

$$\hat{q}_{t+1}\kappa^E - r(x(\omega) - S^e) - e \geq rS^e \Longleftrightarrow \omega \leq \bar{\omega}\left(\hat{q}_{t+1}\right).$$

By assumption $\tilde{\omega}\left(\hat{q}_{t+1}, S^e\right) \leq \bar{\omega}\left(\hat{q}_{t+1}\right)$, so the entrepreneur with $\omega \leq \tilde{\omega}\left(\hat{q}_{t+1}, S^e\right)$ chooses to invest in the project when facing these two options.

(b) The contract that induces no effort is feasible iff

$$\sum_i \left(\hat{q}_{t+1}\kappa_i - c_i\right) \Pr(i|a=0) = r(x(\omega) - S^e)$$

and $0 \leq c_2 \leq c_1$. That is, the contract that induces no effort is feasible iff $\omega \leq w\left(\hat{q}_{t+1}, S^e\right)$ with

$$\hat{q}_{t+1}\left((1-\pi)\kappa_1 + \pi\kappa_2\right) = r\left(x\left(w\left(\hat{q}_{t+1}, S^e\right)\right) - S^e\right)$$

An entrepreneur with $\omega \leq w\left(\hat{q}_{t+1}, S^e\right)$ has two options: (i) invest in the project (with the optimal contract that induces no effort), with expected payoff $\pi c_2 = \hat{q}_{t+1}\sum_i \kappa_i \Pr(i|a=0) - r(x(\omega) - S^e)$ or (ii) invest in savings technology, with return: rS^e. Hence, the entrepreneur chooses the project iff

$$\hat{q}_{t+1}\left((1-\pi)\kappa_1 + \pi\kappa_2\right) - r(x(\omega) - S^e) \geq rS^e \Longleftrightarrow \omega \leq \hat{\omega}\left(q_{t+1}\right)$$

with

$$\hat{q}_{t+1}\left((1-\pi)\kappa_1 + \pi\kappa_2\right) = rx\left(\hat{\omega}\left(q_{t+1}\right)\right). \tag{10}$$

Note that $\hat{\omega}\left(q_{t+1}\right)$ does not depend on S^e for the same reason $\bar{\omega}\left(\hat{q}_{t+1}\right)$ does not depend on S^e.

(c) From (9) and (10), we have:

$$rx\left(\bar{\omega}\left(\hat{q}_{t+1}\right)\right) = \hat{q}_{t+1}\left((1-\pi^E)\kappa_1 + \pi^E\kappa_2\right) - e$$
$$rx\left(\hat{\omega}\left(q_{t+1}\right)\right) = \hat{q}_{t+1}\left((1-\pi)\kappa_1 + \pi\kappa_2\right)$$

Assuming (2), we have

$$rx\left(\bar{\omega}\left(\hat{q}_{t+1}\right)\right) - rx\left(\hat{\omega}\left(q_{t+1}\right)\right) > 0 \Longleftrightarrow \bar{\omega}\left(\hat{q}_{t+1}\right) > \hat{\omega}\left(q_{t+1}\right).$$

Now suppose $\hat{\omega}\left(\hat{q}_{t+1}\right) < \tilde{\omega}\left(\hat{q}_{t+1}, S^e\right)$. Then, we have the following optimal decisions

i. For $\omega > \tilde{\omega}\left(\hat{q}_{t+1}, S^e\right)$, the entrepreneur does not accept any contract and puts funds in storage.

ii. For $\omega \in (\hat{\omega}(\hat{q}_{t+1}), \tilde{\omega}(\hat{q}_{t+1}, S^e)]$, the entrepreneur prefers the optimal effort-inducing contract rather than the storage, and prefer the storage rather than the optimal non-effort-inducing contract. Thus, entrepreneurs take up the effort-inducing contract.

iii. For $0 \leq \omega \leq \hat{\omega}(\hat{q}_{t+1})$, the entrepreneur prefers either optimal contracts with effort or no effort over the savings technology. Both contracts are feasible. The optimal effort-inducing contract is preferable when (2) holds.

(d) Consider the case in which making effort is socially efficient, i.e., $\hat{\omega}\left(q_{t+1}\right) < \tilde{\omega}\left(\hat{q}_{t+1}, S^e\right)$. Then, the supply of capital is determined by

$$k_{t+1} = \eta \tilde{\omega}\left(\hat{q}_{t+1}, S^e\right) \kappa^E.$$

The key difference with the economy in (1), $k_{t+1} = \eta \bar{\omega}\left(\hat{q}_{t+1}\right) \kappa^E$, is that now supply depends on today's savings so we can have lasting effects of productivity shocks, even though productivity shocks themselves are i.i.d. Note that it may be the case that savings become low enough such that the unique equilibrium has a low $\hat{q}_{t+1}$, making effort not worthwhile. Under B-G (hidden outcomes), the capital supply curve is given

$$k_{t+1} = \eta \kappa^E \overline{\omega}\left(\hat{q}_{t+1}\right) - h_t(\hat{q}_{t+1}, S^e)$$

where $h_t(\hat{q}_{t+1}, S^e)$ is the expected total auditing cost. One difference is that S^e affects the threshold of investment productivity in the model with moral hazard, while in the BG model the key amplification mechanism operates through the dependence of auditing costs on S^e.

The effects of a positive productivity shock θ are similar to the effects in the BG model. The main differences are the channels through which S^e affects aggregate investment.