14.454 (Long) Problem Set
Q5-Q6 Solutions

1 Bank runs and contagion

Diamond-Dybvig model (1983). First, suppose all π_i are equal to π, so all regions are identical.

1. **Equilibrium without financial markets.** Agents in autarky solve

$$\max_I \pi \ln(1 - I + LI) + \beta(1-\pi)\log(1 - I + RI)$$

In an interior optimum, the first-order condition gives

$$\frac{\pi}{1 - I + LI}(1 - L) = \frac{\beta(1-\pi)}{1 - I + RI}(R - 1)$$

Solving yields $I = \frac{1}{\pi+\beta(1-\pi)}(\frac{\beta(1-\pi)}{1-L} - \frac{\pi}{R-1})$. If this formula is below 0, the optimum is at $I = 0$; if above 1, the optimum is $I = 1$. As $L \to 1, I \to 1$, whereas if $R \to 1$ and L is away from 1, we have $I \to 0$.

2. **Equilibrium with bond markets.** Under the bond market, the consumptions are $c_1 = (1 - I) + RIp$ and $c_2 = \frac{1-I}{p} + RI$. Bond market clearing requires $\pi c_1 = (1 - I)$ and $(1 - \pi)c_2 = RI$. Moreover, considering the period 0 investment decision, we must have $p = 1/R$; if $p < 1/R$, nobody would invest in the short asset; if $p > 1/R$, nobody would invest in the long asset. Thus $p = 1/R$, $I = 1 - \pi$, and solving, we have $c_1 = 1$ and $c_2 = R$. This equilibrium is clearly better than autarky because resources do not go to waste and it is ex-post efficient, but it is not good in the sense that it is ex-ante inefficient - in period 0, this does not provide sufficient insurance (as will be seen in part 3).

3. **Social planner's allocation.** The planner solves

$$\max \pi \ln(c_1) + \beta(1-\pi)\ln(c_2)$$

subject to

$$c_1 = \frac{1}{\pi}(1 - I),\ c_2 = \frac{1}{1-\pi}RI, \text{ and } c_2 \geq c_1.$$

Solving the problem without the IC, we have $(c_1^*, c_2^*) = (\frac{1}{\pi+\beta(1-\pi)}, \frac{\beta R}{\pi+\beta(1-\pi)})$. Since $\beta R > 1$, we get $c_2^* > c_1^*$, so the IC constraint does not bind.

4. **Equilibrium with banks.**

(a) Impatient individuals get nothing if they do not withdraw at $t = 1$ because they do not value consumption at period 2. Thus they all withdraw.

(b) Note that λ is a sufficient statistics for the consumers' payoffs. Let $\{c_1(\lambda), c_2(\lambda)\}$ denote the payoffs when withdrawing at $t \in \{1,2\}$, and denote action $a \in \{0,1\}$ where 1 corresponds to withdrawing early. When $\lambda = \pi$. There is no liquidation in this case. Thus, $c_1(\lambda) = c_1^*$ and $c_2(\lambda) = c_2^*$. Let $\lambda > \pi$. There is liquidation in this case. Let M denote the amount of investment left after liquidation. Thus,

$$M = 1 - \left(\pi + \frac{\lambda - \pi}{L}\right) c_1^*$$

Therefore,

$$c_1(\lambda) = \min\left\{\frac{\pi c_1^* + L(1 - \pi c_1^*)}{\lambda}, c_1^*\right\}$$

$$c_2(\lambda) = \max\left\{\frac{1 - \left(\pi + \frac{\lambda-\pi}{L}\right) c_1^*}{1 - \lambda} R, 0\right\}$$

The best response of a patient agent satisfies:

$$\hat{a}^P(\lambda) \begin{cases} = 1 & \text{if } c_1(\lambda) > c_2(\lambda) \\ \in \{0,1\} & \text{if } c_1(\lambda) = c_2(\lambda) \\ = 0 & \text{otherwise} \end{cases}$$

given the storage technology at period $t = 1$.

(c) Find all the Nash equillibria at $t = 1$. How does the set of equilibria depend on β? Note that:

$$c_1^* \equiv c_1(\pi) < c_2(\pi) \equiv c_2^*$$

and

$$c_1(\lambda) = c_1^* \Leftrightarrow \lambda \leq \bar{\lambda} \equiv \pi + L\left(\frac{1}{c_1^*} - \pi\right) = \pi + L\beta(1 - \pi) < 1$$

since $\beta L < 1$, and $c_1(\cdot)$ is strictly decreasing if $\lambda \in (\bar{\lambda}, 1]$. Furthermore, note that

$$\lambda \geq \bar{\lambda} \Rightarrow c_2(\lambda) = 0$$

and $c_2(\cdot)$ is strictly decreasing since

$$\frac{\partial}{\partial \log(\lambda)} \log\left(1 - \left(\pi + \frac{\lambda - \pi}{L}\right) c_1^*\right) < \frac{\partial}{\partial \log(\lambda)} \log(1 - \lambda)$$

since $L \leq 1 < c_1^*$. Therefore, there exists $\tilde{\lambda} \in (\pi, \bar{\lambda})$ such that $c_1(\tilde{\lambda}) = c_2(\tilde{\lambda}) > 0$. Thus,

$$\tilde{\lambda} = \pi + \frac{L}{1 - L}\left[\frac{1}{c_1^*} - 1\right]$$

Therefore, there are three (pure strategy) Nash equilibria:

- Equilibrium A (no run): $a^I = 1, a^P = 0 \Rightarrow \lambda = \pi$
- Equilibrium B (partial run): $a^I = 1$, $a^P = 1$ for mass $\tilde{\lambda} - \pi$ and $a^P = 0$ for mass $1 - (\tilde{\lambda} - \pi) \Rightarrow \lambda = \tilde{\lambda}$
- Equilibrium C (full run): $a^I = 1, a^P = 1 \Rightarrow \lambda = 1$

The set of equilibria are the same, but the probability of run in equilibrium B changes with β.

5. **Policy intervention.**

   (a) **Deposit insurance financed by taxes at $t = 2$.** With credible deposit insurance, the efficient allocation can still be implemented. In the no-run equilibrium, only impatient consumers withdraw at $t = 1$, so the bank implements (c_1^*, c_2^*) exactly. The run equilibrium does not survive if the insurance is credible. Patient consumers no longer need to withdraw early to protect themselves, because their promised payoff is insured. Thus waiting is optimal.

   Credibility requires fiscal capacity. The government must be able to raise enough resources through $t = 2$ taxes, or equivalently issue credible claims backed by future taxes, to honor insured deposits if withdrawals exceed the normal level. In the deterministic π case, the insurance need not be used on the equilibrium path, but it must be credible off path.

   (b) **Suspension of convertibility.** Suspension can also implement the efficient allocation if the suspension threshold is set at the normal impatient withdrawal share, $\lambda^s = \pi$, or just at the maximal normal withdrawal level. On the equilibrium path, only impatient consumers withdraw, so suspension does not bind and the efficient allocation is implemented. The run equilibrium does not survive. Once withdrawals reach the threshold, additional patient depositors cannot withdraw early. Therefore, a patient depositor has no strict incentive to join a run.

   Credibility requires an enforceable legal or institutional commitment to stop withdrawals once the threshold is reached. The bank must be able to deny convertibility to late withdrawers and preserve the remaining long assets for patient consumers.

   (c) **Lender of last resort lending against the long asset.** A lender of last resort can implement the efficient allocation if it can lend enough date 1 liquidity against the bank's long asset. In a panic, instead of liquidating the long asset at value L, the bank borrows against its date 2 return R. The run equilibrium does not survive if the lending facility is fully credible and sufficiently large. Patient consumers know that the bank will not be forced into inefficient liquidation merely because many depositors withdraw early.

   The collateral condition is that the date 2 value of the pledged long asset is sufficient to repay emergency lending. To withstand a full withdrawal attempt without liquidating, the bank needs additional date 1 liquidity equal to $(1 - \pi)c_1^*$. The long asset produces $RI^* = (1 - \pi)c_2^*$. Thus, the collateral condition holds as $c_2^* \geq c_1^*$. More generally, the lender of last resort must either have real date 1 resources or issue claims accepted as liquidity, and the bank's long asset must be sufficient collateral.

Allen-Gale model (2000). Now π_i can take one of the values $\omega_H > \omega_L$, and the realization depends on a state $S \in \{S_1, S_2\}$ with equal probability: (i) When $S = S_1$, $\pi_i = \omega_H$ iff i is odd, and (ii) When $S = S_2$, $\pi_i = \omega_H$ iff i is even.

6. **Social planner's allocation.** Note that all regions are ex-ante identical. The social planner (with equal Pareto weights) thus solves

$$\max \left[N(\omega_H + \omega_L)\ln(c_1) + (2N - N(\omega_H + \omega_L))\beta \ln(c_2)\right]$$

   subject to budget constraints and the incentive compatibility constraint $c_1 \leq c_2$. Divide by $2N$, the system is exactly identical to the problem in parts 1-3 with $\pi = (\omega_H + \omega_L)/2 = \gamma$.

7. **Equilibrium without interbank market.** For any region i, because of the uncertainty of whether $\pi_i = \omega_H$ or ω_L, the region cannot independently provide a deposit contract that replicates the social planner's optimum in both states (it either satisfies for at most one, or neither, if they try to be "correct on average").

   There is no overall shortage of liquidity because if the regions were interconnected, the banks would mimic the social planner's optimum for $\pi = \gamma$; it's that they are distributed unevenly across regions, so each region cannot independently provide liquidity.

8. **Equilibrium with interbank market.** Suppose that banks are allowed to exchange deposits in $t = 1$.

   (a) (Complete markets) Each bank holds $z = (\omega_H - \gamma)/N$ deposits in every other region. Consider a high-liquidity region. It has local date 1 withdrawal demand $\omega_H c_1^*$. It also owes deposits to the $N = 1$ other high-liquidity regions, since those banks also redeem their claims at date 1. Hence its date 1 liabilities are $[\omega_H + (N-1)z]c_1^*$. Its date 1 liquid resources are its storage γc_1^* plus claims on all $2N - 1$ other banks: $[\gamma + (2N-1)z]c_1^*$. The budget condition is

$$\omega_H + (N-1)z = \gamma + (2N-1)z \Longrightarrow Nz = \omega_H - \gamma.$$

   This holds by construction. Now consider a low-liquidity region. It has local date 1 demand $\omega_L c_1^*$ and must repay the N high-liquidity regions that redeem deposits. Thus its date 1 liabilities are $[\omega_L + Nz]c_1^*$. Since $z = (\omega_H - \gamma)/N$, $\omega_L + Nz = \gamma$. So its storage γc_1^* is exactly enough. The date 2 constraints also hold. High-liquidity regions have fewer late consumers but owe claims to low-liquidity regions; low-liquidity regions have more late consumers but receive claims from other regions. The interbank claims reverse the date 1 transfers. Thus the first best is implemented without additional liquidation.

   (b) (Incomplete markets) Each bank i holds $z = \omega_H - \gamma$ deposits in bank $i + 1$, with bank $2N$ holding deposits in bank 1. In state S_1, odd regions have high liquidity demand

and even regions have low liquidity demand. In state S_2, the reverse holds. Because the network alternates, every high-liquidity region holds deposits in a low-liquidity region. For a high-liquidity region i, the date 1 liquidity shortfall is $(\omega_H - \gamma)c_1^*$. It redeems z deposits in region $i+1$, so it obtains $zc_1^* = (\omega_H - \gamma)c_1^*$.

For a low-liquidity region $i+1$, local date 1 demands is $\omega_L c_1^*$, and it must repay zc_1^* to the high-liquidity region i. Its total date 1 demand is

$$(\omega_L + z)c_1^* = \gamma c_1^*,$$

which exactly equals its storage. Thus the incomplete ring also implements the first best in normal states.

9. **Contagion.**

(a) The buffer $b(\omega)$ of a bank is the maximum assets that can be liquidated and made available for consumers in period 1 without causing a run. We need $c_2 \geq c_1^*$for patient consumers to not run; this implies a buffer of

$$b(\omega) = L[x - \frac{(1-\omega)c_1^*}{R}]$$

where x is the investment the bank made in the long asset.

(b) Suppose region 1 falls. The loss imposed on any other region is proportional to $z(c_1^* - q_1)$ where $z = \frac{\omega_H - \gamma}{N}$ and q_1 is the liquidation value of deposits in region 1. When N is large enough, we argue that we have other regions are insolvent but not bankrupt. For any fixed ϵ with $\gamma + \epsilon < 1$, there exists N large enough such that

$$z(c_1^* - q_1) < b(\gamma).$$

Then the other regions may suffer a small loss, but the loss is smaller than their buffer. They do not go bankrupt. Hence the shock is contained in region 1. Indeed, as $N \to \infty$, $z \to 0$; the shock from region 1 will be distributed evenly across other banks.

(c) The two conditions are

$$\epsilon c_1^* > b(\gamma + \epsilon)$$
$$z(c_1^* - \bar{q}_1) > b(\gamma)$$

The first condition implies that region 1 bankrupts as a result of the $\gamma + \epsilon$ shock. The second condition means that region $2N$ bankrupts as a result of region 1 demanding liquidity. These two are sufficient for the entire system to break down; region $2N$ demanding liquidity to region $2N-1$ will trigger bankruptcy (by the second inequality), and so forth.

2 Diamond-Dybvig in a small open economy

Define

$$w \equiv e + \frac{R-1}{R} f.$$

1. **Constrained efficient allocation**

   (a) These constraints are: (6) Investment cannot exceed domestic endowment plus initial foreign borrowing; (7) Impatient consumption at $t = 1$ is financed by new foreign borrowing plus liquidation; (8) Patient consumption plus repayment of foreign debt must be financed by the remaining long asset; (9) Initial foreign borrowing cannot exceed the ceiling; (10) Total foreign borrowing cannot exceed the ceiling; and (11) IC constraint for patient agents, who can imitate impatient agents and store until $t = 2$.

   (b) At the optimum, $l^* = 0$, because liquidation is inefficient and there is not aggregate uncertainty. Constraints (7) and (10) bind, $\pi c_1 = b$ and $d + b = f$. The investment constraint also binds $k = e + d$. Using the $t = 2$ constraint:

   $$(1-\pi)c_2 + d + b = Rk.$$

   Since $d + b = f$ and $k = e + d = e + f - \pi c_1$, we get $(1-\pi)c_2 + f = R(e + f - \pi c_1)$ so

   $$R\pi c_1 + (1-\pi)c_2 = Re + (R-1)f = Rw.$$

   (c) The planner solves

   $$\max \pi \log c_1 + (1-\pi)\log c_2 \ \text{ subject to: } R\pi c_1 + (1-\pi)c_2 = Rw.$$

   The optimality condition is $c_2 = Rc_1$ and then

   $$c_1^* = w \ \text{ and } \ c_2^* = Rw$$

   and the IC constraint holds since $R > 1$. The remaining objects are

   $$b^* = \pi w, \ d^* = f - \pi w, \ k^* = (1-\pi)w + f/R, \ \text{ and } \ l^* = 0.$$

   The maintained assumption that f is sufficiently large ensures $d^* > 0$.

2. **Equilibrium with demand deposits.** The bank offers $(c_1^*, c_2^*) = (w, Rw)$.

   (a) To repay total foreign debt f at $t = 2$, the bank must leave at least f/R units of the long asset. Hence, $l^+ = k^* - f/R$. Using the solution above:

   $$l^+ = (1-\pi)w.$$

   (b) If all depositors withdraw at $t = 1$, total promised withdrawals are $c_1^* = w$. Available $t = 1$ resources are

   $$b^* + Ll^+ = \pi w + L(1-\pi)w.$$

   Thus, the bank fails if

   $$z^+ = c_1^* - (b^* + Ll^+) = (1-\pi)(1-L)w > 0.$$

   Therefore, a bank run is an equilibrium.

(c) z^+ is the bank's $t = 1$ liquidity shortfall in a full run, assuming it preserves enough long assets to repay foreign creditors.

It is the open economy version of Diamond Dybvig illiquidity: short term promised claims exceed immediately available resources, even though the bank may be solvent if assets are held to maturity.

3. **Foreign creditors and ongoing lending.** Assume that, in a run, no new foreign lending b^* is provided.

(a) If the bank must still repay initial foreign debt d^*, it must leave d^*/R units of the long asset. Hence $I^a = k^* - d^*/R$.

(b) In a run, available liquidity is only LI^a. The bank fails if

$$z^a = c_1^* - LI^a = w - L(k^* - d^*/R) > 0.$$

(c) Since $f = d^* + b^*$,

$$z^a - z^+ = b^*(1 - L/R) > 0 \Longrightarrow z^a > z^+.$$

No ongoing foreign lending makes the bank strictly more fragile.

(d) If $z^a > 0 > z^+$, then a run is possible only when foreign creditors refuse new lending at $t = 1$. The domestic bank run is therefore triggered by foreign creditor panic, not by domestic illiquidity alone.

(e) If the bank borrows the full f at $t = 0$, invests part in the long asset, and holds b^* as liquid reserves, then it does not need new foreign lending at $t = 1$. So it is not vulnerable to foreign creditors' confidence about ongoing lending. It may still be vulnerable to a domestic depositor run, but not to the specific rollover or new lending panic in part 3.

4. **Short term external debt.** Now d^* comes due at $t = 1$ and must be rolled over.

(a) If foreign creditors refuse to roll over and all depositors run, $t = 1$ obligations are $c_1^* + d^*$. If the bank liquidates the entire long asset, available resources are Lk^*. Hence the bank fails if

$$z^b = c_1^* + d^* - Lk^* > 0.$$

(b) Assuming that d^* is nonnegative, $z^b > z^a > z^+$. That is to say, under the assumption in this subsection, financial fragility is greatest if lenders refuse to roll over short-term debt in the vent of run.

Short term external debt is most fragile because it adds foreign debt repayment to $t = 1$ obligations. If creditors refuse to roll over, the bank must satisfy both domestic withdrawals and foreign repayment immediately. This makes a self fulfilling foreign creditor panic especially damaging.