14.454 (Long) Problem Set Solutions

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1 RE Bubbles with Incomplete Markets

1. **Benchmark: No bubbles**

   (a) The households' problem (generation t) is:

$$\max \beta \log(c_t^y) + (1-\beta)\log(c_t^o)$$
$$\text{s.t. } c_t^y + K_{t+1} = W_t$$
$$c_t^0 = (1 + r_{t+1})\,K_{t+1}$$

Thus,

$$c_t^y = \beta W_t \text{ and } c_t^o = (1-\beta)(1+r_{t+1})W_t$$

and

$$K_{t+1} = (1-\beta)W_t \tag{1}$$

The demand for labor and capital satisfy:

$$w_t = (1-\alpha)\left(\frac{K_t}{A_t}\right)^{\alpha} \tag{2}$$
$$r_t = \alpha\left(\frac{K_t}{A_t}\right)^{\alpha-1} \tag{3}$$

since $L_t = L \equiv 1$. The resource constraint is:

$$c_t^y + c_{t-1}^0 + K_{t-1} - K_t = F(K_t, A_t) = r_t K_t + W_t \tag{4}$$

By Walras' Law, (4) is redundant. The equilibrium is entirely characterized by (2)-(4). In efficiency units, the capital supply is

$$k_{t+1} = \frac{1-\beta}{1+g} w_t.$$

Replacing this expression for k_{t+1} back into the demand for capital we obtain the desired relationship,

$$r_{t+1} = \alpha\left(\frac{(1-\beta)w_t}{1+g}\right)^{\alpha-1}. \tag{5}$$

(b) The FOC with respect to labor yields

$$w_{t+1} = (1-\alpha)k_{t+1}^{\alpha}.$$

Using the FOC with respect to capital (2),

$$w_{t+1} = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} r_{t+1}^{-\frac{\alpha}{1-\alpha}}.$$

Combining this equation together with (5),

$$r_{t+1} = \left(\frac{\alpha(1+g)}{(1-\beta)(1-\alpha)} \right)^{1-\alpha} r_t^{\alpha}.$$

(c) Let $\psi \equiv \frac{\alpha(1+g)}{(1-\beta)(1-\alpha)}$. Taking logs,

$$\log(r_{t+1}) = (1-\alpha)\log\psi + \alpha\log(r_t)$$

Since $\alpha < 1$, the system is asymptotically stable with long run value $r^* = \psi$. The economy is dynamically inefficient if $\psi < g$.

2. **Adding bubbles**

(a) The only difference is that now savings are dedicated not only to investment in capital but also in bubbles. Thus,

$$s_t = b_t + (1+g)k_{t+1}$$

Using the fact that $s_t = (1-\beta)w_t = (1-\beta)(1-\alpha)k_t^{\alpha}$ and rearranging yields the first equation,

$$(1+g)k_{t+1} + b_t = (1-\beta)(1-\alpha)k_t^{\alpha}.$$

Next, note that the return on the bubble $1 + r^B \equiv \frac{b_{t+1}(1+g)}{b_t}$ must be the same as the return on the capital stock $1 + r_{t+1}$. Otherwise agents would like to invest 0 or an infinite amount in the bubble. Rearranging we obtain the second equation,

$$b_{t+1} = \frac{1 + r_{t+1}}{1+g} b_t.$$

(b) Given k_0, we know that in a bubbleless economy, $\{r_t\}_t \to \bar{r}$. Next, we show that for any initial bubble $b_0 > 0$ and given the same k_0, $r_t(b_0) > r_t(0)$ $\forall t$. First, note that the feasibility constraint implies $k_1(b_0) < k_1(0)$. This implies $r_1(b_0) > r_1(0)$, which in turn implies $b_1 > 0$. Since $k_1(b_0) < k_1(0)$, savings are lower in the bubble economy at date 1. Thus, $k_2(b_0) < k_2(0)$. By induction it follows that $k_{t+1}(b_0) < k_{t+1}(0)$ $\forall t$ (capital accumulation is smaller in the bubble economy). This in turn implies $r_{t+1}(b_t) > r_{t+1}(0)$ $\forall t$. Since $r_t(0) \to \bar{r} > g$, for t large enough, $r_t(0) > g$ and, by our previous result, $r_t(b_0) > g$. This, in turn, implies that the bubble grows at a faster rate than the endowment, which in turn implies that we would violate the feasibility constraint of the economy in finite time (k_{t+1} would need to be negative in order to satisfy feasibility). Hence, in an economy with $r > g$, there can be no bubble.

(c) To draw the phase diagram, note that:

$$\Delta b = 0 \Longrightarrow r_{t+1} = g \Rightarrow k_{t+1} = \left(\frac{\alpha}{g}\right)^{1-\alpha} \Longrightarrow b_t = (1-\beta)(1-\alpha)k_t^{\alpha} - (1+g)\left(\frac{\alpha}{g}\right)^{1-\alpha}$$
$$\Delta k = 0 \Longrightarrow b_t = (1-\beta)(1-\alpha)k_t^{\alpha} - (1+g)k_t$$

Furthermore, when b is large there are fewer resources for investment so k decreases (the $k-$arrows pointing left). When k is large, r is small so the bubble contracts (the $b-$arrows pointing down). Figure 1 shows the phase diagram for some parameter values as well as the saddle-path that leads to $r^* = g$.

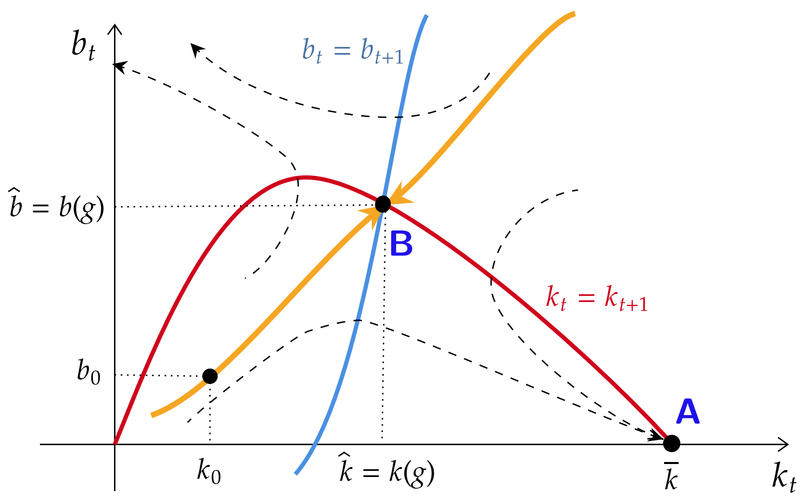

Figure 1: Phase diagram

(d) Using the same argument as in item b), $b_0^{'} < b_0 \Rightarrow \frac{b_1^{'}}{b_1} < \frac{b_0^{'}}{b_0}$ and so forth. Thus, $\frac{b_t^{'}}{b_t}$ does not converge to one, implying there can at most be one path leading to a bubble of non-zero size in per capita terms. It follows from our phase diagram that the value of b_0 consistent with the bubbly steady state increases with k_0.

(e) Using the same argument as in item b), we can show that if $b_0^{'} < b_0$ and $k_0^{'} = k_0$, then $k_{t+1}(b_0^{'}) > k_{t+1}(b_0)$. In this sense, bubbles "crowd-out" investment and this reduces overaccumulation of capital. In a dynamically inefficient economy, this helps solve the inefficient problem.

3. **Endogenous Growth.** Allow now for endogenous growth:

$$A_t = \mu K_t \tag{6}$$

for some $\mu > 0$. Let $1 + g_t \equiv \frac{A_t}{A_{t-1}}$.

(a) (0.5 points) The equilibrium is entirely characterized by:

$$(1+g_{t+1})k_{t+1}+b_t=(1-\beta)(1-\alpha)k_t^{\alpha} \tag{7}$$

$$b_{t+1}=\frac{1+r_{t+1}}{1+g_{t+1}}b_t \tag{8}$$

with

$$r_{t+1}=\alpha k_{t+1}^{\alpha-1} \tag{9}$$

and

$$1+g_t=\frac{A_t}{A_{t-1}}\ ,\ A_t=\mu K_t \tag{10}$$

for some initial condition K_0 . From (7)-(10),

$$\frac{K_{t+1}}{K_t}\frac{1}{\mu}+b_t=(1-\beta)(1-\alpha)\left(\frac{1}{\mu}\right)^{\alpha} \tag{11}$$

and

$$b_{t+1}=\frac{1}{\mu}\frac{1+\alpha\left(\frac{1}{\mu}\right)^{\alpha-1}}{(1-\beta)(1-\alpha)\left(\frac{1}{\mu}\right)^{\alpha}-b_t}b_t \tag{12}$$

for some initial condition K_0. The dynamic system (12) admits two steady states: $b^*=0$ and $\hat{b}^*=[(1-\beta)(1-\alpha)-\alpha]\mu^{-\alpha}-\mu^{-1}$. The former is stable, while the latter is unstable. Note that $b_t\leq\hat{b}^*$ at equilibrium, for each $t\geq 0$. Assume (by contradiction) there exists some $t\geq 0$ such that $b_t>\hat{b}^*$. From (12), $b_t\to\infty$ so that (11) does not hold with $K_t\geq 0$, for each $t\geq 0$. This leads to the desired contradiction.

(b) (0.5 points) Consider two initial values b_0 and b_0', with $0\leq b_0'<b_0\leq\hat{b}^*$. From (12), $b_t'<b_t$, for each $t\geq 0$. From (11), $K_{t+1}<K_{t+1}'$, for each $t\geq 0$. From (6), $A_{t+1}<A_{t+1}'$, for each $t\geq 0$. Therefore, the sequence of output is lower for b_0 than for b_0'.

Note that bubbles can emerge even if the economy is dynamically efficient, from (12), in contrast with Tirole (1985). Bubbles are potentially beneficial because they provide a store of value and implement intergenerational transfers in an economy where the growth rate may exceed the return on capital. However, unlike in the exogenous growth model, they also lower endogenous productivity growth by crowding out capital. Hence there is a genuine tradeoff, and dynamic inefficiency alone does not guarantee that bubbles are welfare improving.

2 Credit Booms

This problem is adapted from Section 1 of Kurlat (2021) - "Investment Externalities in Models of Fire Sales" (JME). No uncertainty and no financial contracts. Preferences are given by $u=$

$c_0 + c_1 + c_2$ for both households and entrepreneurs. Entrepreneurs are endowed with n goods at $t = 0$, and households are endowed with e_0 and e_1 goods at period 0 and 1, respectively. It is not possible to borrow or lend, but there is still a competitive market for maintained capital at $t = 1$ with a price q.

1. The economy's resource constraint are:

   (a) At $t = 0$, the aggregate endowment is composed of entrepreneur's endowment n and household's endowment e_0. The total demand is the sum of the aggregate consumption $c_0^E + c_0^H$ and the level of capital the entrepreneurs convert from the perishable good. Then, resource constraint states that aggregate demand cannot be higher than aggregate endowments:
   $$c_0^E + c_0^H + k \leq n + e_0 \tag{13}$$

   (b) At $t = 1$, the aggregate endowment is e_1 (no production yet). Given capital k, entrepreneurs choose the fraction of capital that will be maintained, αk, and pay maintenance costs, $z\alpha k$. Moreover, the aggregate consumption is $c_1^E + c_1^H$. Then, resource constraint states that aggregate uses cannot exceed aggregate endowment:
   $$c_1^E + c_1^H + z\alpha k \leq e_1 \tag{14}$$

   (c) At $t = 1$, the aggregate demand of capital $k^E + k^H$ cannot be higher than the stock of maintained capital αk, i.e.,
   $$k^E + k^H \leq \alpha k \tag{15}$$

   (d) Finally, at $t = 2$ the aggregate demand $c_2^E + c_2^H$ cannot exceed the aggregate production, Ak^E (from entrepreneur's investments) plus $F(k^H)$ (from household's investments):
   $$c_2^E + c_2^H \leq Ak^E + F(k^H) \tag{16}$$

   Therefore, α can be interpreted as the fraction of capital maintained by the entrepreneur at $t = 1$.

2. The entrepreneurs take price q as given and solve the following problem:
   $$\max_{c_0^E, c_1^E, c_2^E, k, s, k^E, \alpha} \quad c_0^E + c_1^E + c_2^E \tag{17}$$

   subject to

   $$c_0^E + k \leq n \tag{18}$$
   $$c_1^E + z\alpha k \leq sq \tag{19}$$
   $$k^E \leq \alpha k - s \tag{20}$$
   $$c_2^E \leq Ak^E \tag{21}$$

   and $c_t^E, k, k^E \geq 0$; $\alpha \in [0, 1]$; and $s \in [0, \alpha k]$. (18) is the period 0 budget constraint: demand for consumption and capital cannot exceed the entrepreneur's endowment. (19) is the period

1 budget constraint: resources needed (consumption plus maintenance costs) cannot exceed the revenue from selling s units of capital at price q. (20) states that capital carried onto period 2 is not larger than the maintained capital αk minus sold capital s in period 1. Finally, (21) says that consumption in period 2 is lower than the revenue generated from investments. What does s represent? It is the units of capital sold in the capital market in period 1.

3. Households take price q as given and solve the following problem:

$$\max_{c_0^H, c_1^H, c_2^H, k^H} c_0^H + c_1^H + c_2^H \tag{22}$$

subject to

$$c_0^H \leq e_0 \tag{23}$$
$$c_1^H + qk^H \leq e_1 \tag{24}$$
$$c_2^H \leq F(k^H) \tag{25}$$

A competitive equilibrium is a consumption allocation $\left(c_0^i, c_1^i, c_2^i\right)_{i\in\{E,H\}}$, capital allocation $\left(k, \left(k^i\right)_{i\in\{E,H\}}, \alpha, s\right)$, and price of capital q such that

(a) Given q, entrepreneurs solve problem (17) and households solve problem (22).

(b) Capital market clears: $k^E + k^H = \alpha k$.

4. Assume $A > 1 + z$, $F'(0) \in \left(\frac{Az}{A-1}, A\right)$, $-F''(x)x/F'(x) < 1$, $F'\left(\frac{A-1}{A}n\right) < \frac{Az}{A-1}$, and $e_1 > zn$.

(a) Given q, the problem for the entrepreneur in period 1 who holds k units of capital is

$$V(k,q) = \max_{c_1^E, c_2^E, k^E, s, \alpha} c_1^E + c_2^E \tag{26}$$

subject to

$$c_1^E + z\alpha k \leq sq \tag{27}$$
$$k^E \leq \alpha k - s \tag{28}$$
$$c_2^E \leq Ak^E \tag{29}$$

Since $A > 1 + z$, (28) and (29) bind and, then,

$$c_1^E + k^E q \leq \alpha k(q - z) \text{ and } c_2^E = Ak^E.$$

Since the maximum possible value for q is $F'(0) < A$, assumptions above imply $A > q$ and $\alpha = 1$, $c_1^E = 0$, $c_2^E = Ak\frac{q-z}{q}$, and $k^E = k\frac{q-z}{q}$. Thus, $V(k,q) \equiv A\frac{q-z}{q}k$.

(b) (17) is simplified to

$$\max_{c_0^E, k} c_0^E + V(k,q) \text{ subject to: } c_0^E + k \leq n \text{ and } 0 \leq c_0^E.$$

Note that if $A(q-z) > q$, the optimal solution is $k = n$. Otherwise, $k = 0$. Then, the optimal level of capital is

$$k(q,n) = \begin{cases} n & q > \frac{Az}{A-1} \\ \text{any } k \in [0,n] & q = \frac{Az}{A-1} \\ 0 & \frac{Az}{A-1} > q \end{cases}.$$

(c) For (22), all constraints bind at equilibrium and, thus, the problem can be rewritten as

$$\max_{k^H} e_0 + e_1 + F(k^H) - qk^H.$$

The first order condition is given by $F'(k^H) = q$. From capital market clearing condition, we have $s = k^H$, then $F'(s) = q$. Since $k^E = k\frac{q-z}{q} \implies s = k - k\frac{q-z}{q} = \frac{zk}{q}$. Then, the implicit equation pinning down the relationship between q and k is

$$F'\left(\frac{zk}{q(k)}\right) = q(k).$$

Since assumptions on F' guarantee interior solutions in part b), $q = Az/(A-1)$ and $k \in (0,n)$ in equilibrium.

(d) The price $q(k)$ is decreasing in k. A higher stock of capital carried by entrepreneurs from $t = 0$ to $t = 1$ leads to a higher supply in the maintained capital market $s = k\frac{z}{q}$. This reduces the price of capital such that households are encouraged to buy more capital.

5. Constrained optimization problem

(a) The planner solves the following problem

$$\max_{k, c_0^H, c_0^E} c_0^E + V\left(k, q(k)\right)$$

subject to

$$c_0^E + c_0^H + k \leq n + e_0,\ \Pi \leq c_0^H + e_1 + F(k^H) - q(k)k^H, \text{ and } \frac{z}{q(k)}k = k^H.$$

Only the period 1 resource constraint is considered as the planner can only make transfers at that period. In addition, we include the equilibrium condition for the maintained capital as it is directly related to $q(k)$.

(b) Let $s\left(q(k), k\right) \equiv zk/q(k)$, then the last inequality can be written in terms of k as $\Pi \leq e_0 + e_1 + F\left(s\left(q(k), k\right)\right) - zk$. Thus, we can rewrite the planner's problem as

$$\max_{k, c_0^H, c_0^E} c_0^E + V\left(k, q(k)\right)$$

subject to

$$c_0^E \leq n + e_0 - k - c_0^H \text{ and } \Pi \leq c_0^H + e_1 + F\left(s\left(q(k), k\right)\right) - zk.$$

Since these conditions bind,

$$\max_k \mathcal{W}(k) \equiv n + e_0 + e_1 - k - \Pi - zk + F\left(s\left(q(k), k\right)\right) + V\left(k, q(k)\right)$$

(c) Let us compute the first-order derivative of the objective function

$$\frac{d\mathcal{W}(k)}{dk} = -(1+z) + \left(z - \frac{kz}{q}q'\right)\frac{F'}{q} + V'_k + V'_q q'$$

We can evaluate this derivative at the competitive equilibrium $F'/q = 1$, $V'_k = A(q - z)/q$, $V'_q = Akz/q^2$, and $q = \frac{Az}{A-1}$. Thus,

$$\frac{d\mathcal{W}(k)}{dk} = \frac{(A-z-1)(A-1)kq'(k)}{Az} < 0$$

Then, reducing k increases welfare, i.e., solution k^* is lower than k (the level at the competitive equilibrium).

(d) In Lorenzoni (2008), the pecuniary externality is the entrepreneurs not internalizing the fact that their investment and borrowing choices affect the price of capital $q(k)$. Here this is the same: the marginal investment will affect other entrepreneurs' constraints and liquidation requirements in future periods, but they do not internalize this in the competitive equilibrium.