14.454 (Long) Problem Set
Solutions

May 10, 2025

1 Liquidity Trap

1. In a flexible price competitive equilibrium, given w_t, households maximize net income

$$\max_n w_t n_t - \frac{(n_t)^{1+\psi}}{1+\psi}$$

leading to $w_t = n_t^{\psi}$ (we ignore superscripts as decisions are symmetric). Zero-profit condition imply $1 = n_t^{\psi} \implies n_t^* = 1$. Let $\tilde{c}_t^h \equiv c_t^h - \frac{1}{1+\psi}$, then, given d_0^h and $\{r_t\}$, each household $h \in \{b, l\}$ solves

$$\max_{\left\{\tilde{c}_t^h, d_{t+1}^h\right\}} \sum_{t=0}^{\infty} \left(\beta^h\right)^t \log \tilde{c}_t^h \text{ subject to:}$$

$$\tilde{c}_t^h = \frac{\psi}{1+\psi} - d_t^h + d_{t+1}^h/(1+r_{t+1}) \text{ and } d_{t+1}^h \leq \phi_{t+1} \text{ for all } t \geq 0.$$

A flexible-price competitive equilibrium is a sequence $\left\{\left\{c_t^h, d_{t+1}^h\right\}_{h\in\{b,l\}}, r_t\right\}_{t\geq 0}$ such that (i) given d_0^h and $\{r_t\}$, $\left\{c_t^h, d_{t+1}^h\right\}$, solves h's problem and (ii) market clears: $d_{t+1}^h + d_{t+1}^l = 0$ for all $t \geq 0$.

Assuming $d_t^b = \underline{\phi}$ for all $t \geq 2$, then the equilibrium allocation for $t \geq 2$ is given by $d_t^l = -\underline{\phi}$, $r_{t+1} = 1/\beta^l - 1$, $c_t^b = 1 - \underline{\phi}(1-\beta^l)$, and $c_t^l = 1 + \underline{\phi}(1-\beta^l)$. For $t = 1$, lenders are unconstrained because $d_2^l < 0$, so the real interest rate is determined by lenders intertemporal decision:

$$\frac{1}{1+r_2} = \beta^l \frac{\tilde{c}_1^l}{\psi(1+\psi)^{-1} + \underline{\phi}(1-\beta^l)} \tag{1}$$

Moreover,

$$\tilde{c}_1^l = \frac{\psi}{1+\psi} + d_1 - \frac{\underline{\phi}}{1+r_2},\ \tilde{c}_1^b = \frac{\psi}{1+\psi} - d_1 + \frac{\underline{\phi}}{1+r_2}, \text{ and } -d_1^l = d_1^b = d_1. \tag{2}$$

Combining (1) and (2) yields the equilibrium allocation as a function of d_1:

$$\frac{1}{1+r_2} = \beta^l \frac{\psi(1+\psi)^{-1} + d_1}{\psi(1+\psi)^{-1} + \underline{\phi}}, \ \tilde{c}_1^l = \Delta_0^l + \Delta_1 d_1, \ \text{ and } \ \tilde{c}_1^b = \Delta_0^b - \Delta_1 d_1 \tag{3}$$

where

$$\Delta_1 \equiv 1 - \frac{\beta^l \underline{\phi}}{\psi(1+\psi)^{-1} + \underline{\phi}}, \ \Delta_0^l \equiv \frac{\psi}{1+\psi}\Delta_1, \ \text{ and } \ \Delta_0^l \equiv 2\frac{\psi}{1+\psi} - \Delta_0^l.$$

For $t = 0$, there are two possibilities. First, if $d_1^b < \overline{\phi}$, both agents are unrestricted, and the competitive equilibrium allocation satisfy

$$\frac{1}{1+r_1} = \beta^l \frac{\tilde{c}_0^l}{\tilde{c}_1^l} = \beta^b \frac{\tilde{c}_0^b}{\tilde{c}_1^b} \tag{4}$$

$$\tilde{c}_0^l = \frac{\psi}{1+\psi} + d_0 - \frac{d_1}{1+r_1} \ \text{ and } \ \tilde{c}_0^b = \frac{\psi}{1+\psi} - d_0 + \frac{d_1}{1+r_1}. \tag{5}$$

Combining (4) and (5) yields an equation that pins down d_1 in equilibrium:

$$\frac{1}{1+r_1} = \beta^l \frac{\psi(1+\psi)^{-1} + d_0}{\Delta_0^l + (\Delta_1 + \beta^l) d_1} = \beta^b \frac{\psi(1+\psi)^{-1} - d_0}{\Delta_0^b - \left(\Delta_1 + \beta^b\right) d_1}. \tag{6}$$

Alternatively, if $d_1^b = \overline{\phi}$, the consumption allocation is given by

$$\tilde{c}_0^l = \frac{\psi}{1+\psi} + d_0 - \frac{\overline{\phi}}{1+r_1} \ \text{ and } \ \tilde{c}_0^b = \frac{\psi}{1+\psi} - d_0 + \frac{\overline{\phi}}{1+r_1},$$

and the real interest rate $(1+r_1)^{-1} = \beta^l \tilde{c}_0^l / \tilde{c}_1^l$. In this case, it is easy to compute allocations for $t = 1$.

2. Assume $\overline{\phi}$ is large enough such that the borrowing constraint does not bind at $t = 0$. From (1),

$$\frac{1}{1+r_2} = \beta^l \frac{\psi(1+\psi)^{-1} + d_1}{\psi(1+\psi)^{-1} + \underline{\phi}}.$$

Therefore, $r_2 < 0$ whenever d_1 is large enough: $d_1 > \psi(1+\psi)^{-1}\left(1/\beta^l - 1\right) + \underline{\phi}/\beta^l$. From (6),

$$\frac{\psi(1+\psi)^{-1} + d_0}{\psi(1+\psi)^{-1} - d_0} = \frac{\beta^b}{\beta^l} \frac{\Delta_0^l + (\Delta_1 + \beta^l) d_1}{\Delta_0^b - \left(\Delta_1 + \beta^b\right) d_1}.$$

Thus, d_1 increases with d_0, but decreases with β^b. Therefore, the real interest rate at $t = 2$ is negative when d_0 is large enough, $d_0 > \overline{d}_0(\beta^b)$, or when β^b is small enough, $\beta^b < \underline{\beta}^b(d_0)$. That is, the borrower at date 0 will leverage too much if they have a sufficient strong motive which is captured by a low discount factor or if they have accumulated a large amount of debt in the past.

3. Assume $\overline{p} = 1$ and firms have pre-committed to satisfy any demand at this price.

   (a) The central bank sets the interest rate according to $r_{t+1} = \max\left\{0, r^*_{t+1}\right\}$. The equilibrium for $t = 2$ onwards is the same as in the part 1. For $t = 1$, the equilibrium depends on whether $d_0 > \overline{d}_0(\beta^b)$ or $\beta^b < \underline{\beta}^b(d_0)$. If neither of this condition holds, then the equilibrium is characterized by the same allocation given in part 1. Instead, if at least one of these conditions hold,

$$r_2 = 0,\ \tilde{c}^b_1 = \tilde{e}_1 - d_1 + \underline{\phi} \text{ and } \tilde{c}^l_1 = \frac{1}{\beta^l}\left(\psi(1+\psi)^{-1} + \underline{\phi}(1-\beta^l)\right)$$

   and

$$\tilde{e}_1 = \frac{\tilde{c}^b_1 + \tilde{c}^l_1}{2} \Longrightarrow \tilde{e}_1(d_1) = \tilde{c}^l_1 + \underline{\phi} - d_1,$$

   so we have a recession.

   (b) Suppose the Central Bank impose a debt ceiling $d_1 \leq \overline{d}$. If the equilibrium does not bind the debt ceiling, i.e., $d_1 < \overline{d}$, this policy intervention will not change the welfare of the agents. However, if it restrict debt (for any initial equilibrium with $d_1 > \overline{d}_1$), it would improve the lifetime welfare of both individuals. In particular, with the debt ceiling set below the ZLB-threshold debt, the CB avoids a recession in period 1.

   (c) Suppose the Central Bank promises a boom from $t = 2$ onwards. The effects of this policy are very different if the central bank announces that policy at $t = 0$ or at $t = 1$. If the announcement is made at $t = 0$, it it may induce even more borrowing, increasing the likelihood to face a crisis. Instead, if the announcement is made in the middle of the crisis at $t = 1$, then it increases $\tilde{c}^l_t$ for $t \geq 2$, inducing a higher interest rate r_2 and avoiding a depression of output (potentially pushing the economy out of the ZLB). The benefits of this change is not uniform across agents: at $t = 1$, lenders are benefited from the increase in interest rate, for $t \geq 2$ the borrowers are benefited from lower interest rates.

4. Macroprudential policies directly limit overborrowing, so the main channel is the reduction of the aggregate demand externality by preventing it, i.e., direct "control" over debt. Instead, "forward guidance" reduces aggregate demand externality by an expectation channel that creates an aggregate demand externality in an opposite direction.

   The main externality here is aggregate demand externality: In the liquidity trap, borrowers' demand depends on current income and it generates a negative multiplier effect that reduces output. Ex-ante interest rate policy is inferior relative to macroprudential policy as it generates a negative income effect on borrowers making recession worse because of a higher leverage. Moreover, the increment in interest rate policy reduces output. The ex-post interest rate policy (forward guidance) is also inferior because it generates redistributive effects, time-inconsistency problems, and relies on the strength of the expectation channel.

2 Bubbles & Capital Flow Volatility

1. **Interpretation.** An entrepeneur borrows I_{t+1} units of international goods and repay $p_{t+1}I_{t+1}$ units of domestic goods. Due to financial frictions, the entrepreneur can only pledge a frac-

tion ψ of their endowment, RK_t. Clearly, this equation rules out using output, RI_{t+1}, as collateral.

2. **Equilibrium.** In equilibrium, the loan markets require that

$$1 \leq p_{t+1} \leq R,$$

since at a price below one, lenders will not lend, and at a price above R borrowers will not borrow. The supply of loans when $p_{t+1} > 1$ is $W_t'/2$ and the constrained demand for loans is $\psi RK_t/2p_{t+1}$. So if $p_{t+1} > 1$, we get

$$\text{Demand} = \frac{\psi RK_t}{2p_{t+1}} < \frac{W_t'}{2} = \text{Supply} \Longrightarrow \text{Excess Supply.}$$

Therefore, in equilibrium we need $p_{t+1} = 1$ (note that we can pin down the equilibrium because lenders will be indifferent between lending and not).

3. **Dynamic inefficiency.** Assume that $W_{t+1} = (1+g)W_t$ where $g > r^*$:

   (a) In this economy,

$$\text{Net Ouflow}_t = W_t - (1+r^*)W_{t-1} = (g - r^*)W_{t-1} > 0.$$

   The economy behaves as a dynamically inefficient economy, in the sense that store of value (rather than the marginal product of capital) has a lower return than the rate of growth of the endowment. In this context, dynamic inefficiency means that the country is lending (or leaving) too many international goods abroad.

   (b) Rather than lending W_t abroad, the young could transfer their W_t endowment of international goods to the old, and when old, receive the W_{t+1} endowment of the new young, and so on. Each generation, effectively, would receive a return $g > r^*$ on its international goods endowment. The OLG assumption impede young agents of different generations to write contracts between themselves, moreover the domestic capital market is segmented since foreign investors cannot participate in it.

4. **Bubbles.** Assume the existence of a bubble ("real estate") B_t that holds no value, but can crash with probability λ.

   (a) The expected return of the bubble is given by

$$\widehat{r}^b = (1-\lambda)g - \lambda = g - \lambda(1+g) < g.$$

   (b) Let $\Delta^b = g - r^*$. For agents to invest in the bubble,

$$\widehat{r}^b - r^* = (1-\lambda)\Delta r^b - \lambda(1+r^*) > 0. \tag{7}$$

   Thus, we can see that this is true when the spread Δr^b is large enough or λ is small enough.

(c) Agents investing a fraction α_t of their wealth in bubbles receive

$$W_t' = W_t(1 + r^* + \alpha_t(\overline{r}_{t+1} - r^*))$$

as a total return of their investment. At $t+1$, an entrepreneur with an investment opportunity enters into two transactions. First, he sells his bubble asset to the next generation to recieve return $\overline{r}_{t+1}$. Next, he borrows l_{t+1} at price $\widetilde{p}_{t+1}$ from bankers, to yield

$$RK_t + RW_t' + (R - \widetilde{p}_{t+1})l_{t+1} \text{ where } l_{t+1} \leq \frac{\psi R}{p_{t+1}} K_t.$$

A banker at date $t+1$ collects international goods by selling his real estate assets to the next generations and then lends these goods, along with any savings from date t internationa lending, to the investing entrepreneurs. As a result, a banker recieve $RK_t + W_t'\widetilde{p}_{t+1}$.

(d) Since ex-ante there is a probability $p = 1/2$ of being an entrepreneur or saver the portfolio choice problem is

$$\max_{0 \leq \alpha_t \leq 1} \mathbb{E}_t \left[RK_t + W_t' \frac{R + \widetilde{p}_{t+1}}{2} + \frac{R - \widetilde{p}_{t+1}}{2} \frac{\psi R}{p_{t+1}} K_t \right].$$

For an interior solution, the condition is

$$(1 - \lambda) \frac{\Delta r^b}{1 + r^*} (R + p_{t+1}^B) - \lambda(R + p_{t+1}^C) = 0 \tag{8}$$

where p_{t+1}^B and p_{t+1}^C represent the equilibrium price of loans when the bubble survives and crashes, respectively.

(e) The supply of funds from bankers is at most W_t', while the demand for funds from entrepreneurs is at most $\frac{\psi R}{p_{t+1}} K_t$. Thus, market clearing in the loan market yields for the no-crash state

$$p_{t+1}^B = \max \left\{ 1, \frac{\psi R}{1 + r^* + \alpha_t \Delta r^b} \right\} = 1$$

since $\psi R < 1$. For the crash state, we will focus on interior solutions. Note that

$$p_{t+1}^C = \max \left\{ 1, \min \left\{ \frac{\psi R}{(1 + r^*)(1 - \alpha_t)}, R \right\} \right\}. \tag{9}$$

Note that if $p_t^C = R$, then it is never optimal to have positive investment in the bubble, so $\alpha = 1$, analogously if $p_t^C = 1$ then $\alpha = 0$. So, we focus on the case where

$$p_t^C = \frac{\psi R}{(1 + r^*)(1 - \alpha_t)}.$$

Combining the above expression with (8), we conclude that

$$\alpha_t = 1 - \frac{\psi R}{1 + r^*} \left[\frac{1 - \lambda}{\lambda} \frac{1 + R}{R} \frac{\Delta r^b}{1 + r^*} - R \right]^{-1}. \tag{10}$$

When $\lambda > 0$, the economy faces a risk-return trade-off: lower capital net outflows versus the higher expected cost of the crash (domestic economy disruption and reversal of capital flows).

5. **Excess volatility**

(a) The total output is given by

$$U = RK_t + W_t' \frac{R + \widetilde{p}_{t+1}}{2} + \frac{R - \widetilde{p}_{t+1}}{2} \frac{\psi R}{\widetilde{p}_{t+1}} K_t.$$

If there is no crash, then

$$U^B = \frac{R+1}{2} W_t (1 + r^* + \alpha_t \Delta r^b) + \frac{R-1}{2} \psi R K_t + RK_t.$$

Consider a small crash scenario where α_t is small enough such that $p^C = 1$, then

$$U^{C,S} = \frac{R+1}{2} W_t (1-\alpha_t)(1+r^*) + \frac{R-1}{2} \psi R K_t + RK_t.$$

Consider now a large crash scenario where α_t is large enough such that $p_t^C = \frac{\psi R}{(1+r^*)(1-\alpha_t)}$, then

$$U^{C,L} = RW_t(1-\alpha_t)(1+r^*) + RK_t.$$

By (9), the threshold is $\alpha^S \equiv 1 - \frac{\psi R}{1+r^*}$. Then, define

$$U^C \equiv \begin{cases} U^{C,L} & \alpha_t \geq \alpha^S \\ U^{C,S} & \text{otherwise} \end{cases}.$$

The planner's problem is then

$$\max_{0 \leq \alpha \leq 1} \lambda U^C + (1-\lambda) U^B.$$

(b) If the solution to this problem, α_t^*, is lower than α^S, then the first-order condition implies

$$W_t \left[(1-\lambda)\Delta r^b - \lambda(1+r^*) \right] \leq 0$$

which is inconsistent with (7). Then, $\alpha_t^* \geq \alpha^S$. Then, the marginal net benefit of increasing α_t is

$$W_t \left[(1-\lambda) \frac{\Delta r^b}{1+r^*} (R+1) - 2\lambda R \right].$$

Comparing this condition with (8), if $p_{t+1}^C < R$,

$$W_t \left[(1-\lambda) \frac{\Delta r^b}{1+r^*} (R+1) - 2\lambda R \right] < 0 \Longrightarrow \alpha_t^* = 1 - \frac{\psi R}{1+r^*}.$$

From (10), $\alpha_t > \alpha_t^*$. This is because of overexposure to risk in this case: "Ex-ante, the low return on lending to entrepreneurs translates into a lack of prudence in the date t portfolio decisions. Bankers chase higher returns by investing excessively in the risky real estate bubble, rather than retaining some international liquidity to be in a position to lend to the entrepreneurial sector. Lack of financial development, in the dimension of tighter domestic collateral constraints, overexpose the economy to the risky bubble." (C-K 2006, page 46).

(c) In the event of a crash, credit crunch raises the price of loans due to an increment in storage needs and a fall in domestic credit. This makes a bubble a riskier asset with larger negative economic consequences.

6. **Capital flow sterilization**

(a) The government will sell bonds today to young agents of generation t and promise to repay them tomorrow by taxing international endowments of generation t tomorrow. By offering a return $r \geq r^b$ the government can "crowd-out" the bubble investment by issuing enough debt to implement the desired α^*. To do that, the government has to be able to credibly at every period the younger generation, and when the bubble is larger this taxation must also be larger to sustain the public debt.

(b) Note that in the previous case there is no transfers between generations, so this is not a solution to the dynamic inefficiency problem. However, if the public debt is supported by credible taxation of future generations then the government can solve both fragility and the dynamic inefficiency. That would be done by setting the interest rate $r = g$ and roll over the debt perpetually, but that will require a strong credibility.