

# 14.471: Fall 2012:PS3 Solutions

December 4, 2012

## 1. Commodity Taxation

Consider an economy consisting of identical individuals with preferences given by

$$U(x_1, x_2, l) = \alpha_1 \log(x_1) + \alpha_2 \log(x_2) + (1 - \alpha_1 - \alpha_2) \log(1 - l).$$

Individuals supply labor,  $l$ , at a market wage of unity, and purchase goods 1 and 2, both of which have fixed producer prices equal to 1. The government's revenue requirement is  $R$ , measured in the fixed producer prices.

**(a) Assume that the government is restricted to using linear commodity taxes on goods 1 and 2 to raise revenue. Set up the Ramsey optimal tax problem for this economy, and find the optimal tax rates on goods 1 and 2 as a function of  $R$  and the individual preference parameters.**

Let us use the primal approach.

The implementability constraint (combining the budget constraint and the FOC) is:

$$\begin{aligned} \frac{\partial U}{\partial x_1} x_1 + \frac{\partial U}{\partial x_2} x_2 + \frac{\partial U}{\partial l} l &= 0 \\ \Rightarrow \frac{\alpha_1}{x_1} x_1 + \frac{\alpha_2}{x_2} x_2 - \frac{1 - \alpha_1 - \alpha_2}{1 - l} l &= 0 \\ &\Rightarrow \alpha_1 + \alpha_2 = l. \end{aligned}$$

Therefore the primal version of the Ramsey problem for this economy is

$$\begin{aligned} \max \quad & \alpha_1 \log(x_1) + \alpha_2 \log(x_2) + (1 - \alpha_1 - \alpha_2) \log(1 - l) \\ \text{s.t.} \quad & l = \alpha_1 + \alpha_2 \\ & x_1 + x_2 + R = l, \end{aligned}$$

where the first constraint is the implementability constraint and the second constraint is the resource constraint. Using the implementability constraint to eliminate  $l$ , the problem can be reduced to

$$\begin{aligned} \max \quad & \alpha_1 \log(x_1) + \alpha_2 \log(x_2) + (1 - \alpha_1 - \alpha_2) \log(1 - \alpha_1 - \alpha_2) \\ \text{s.t.} \quad & R = \alpha_1 + \alpha_2 - x_1 + x_2. \end{aligned}$$

The first order conditions for  $x_1$  and  $x_2$  are

$$\begin{aligned} \alpha_1/x_1 &= \lambda \\ \alpha_2/x_2 &= \lambda. \end{aligned}$$

The resource constraint then implies

$$\lambda = \frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 - R}.$$

Since consumer prices are proportional to marginal utilities and the marginal utilities are equal the consumer prices must be equal. Since the producer prices are also equal, there is a common tax rate for both goods. The government revenue requirement then implies

$$R = t(x_1 + x_2) = t \left( \frac{\alpha_1 + \alpha_2}{\lambda} \right)$$

or

$$t = \frac{R}{\alpha_1 + \alpha_2 - R}.$$

**(b) Obtain expressions for the marginal utility of money of the individual,  $\alpha$ , and the marginal cost of government revenue,  $\lambda$ . How do changes in the government revenue requirement affect these costs and the difference between the two?**

$\alpha$  can be determined from the knowledge that it is equal to the ratio of the marginal utility of any good to the price of that good, or

$$\alpha = \frac{\partial U_1}{\partial x_1} \frac{1}{q_1} = \frac{\alpha_1}{x_1} \frac{1}{1 + t_1} = \lambda \frac{1}{1 + \frac{R}{\alpha_1 + \alpha_2 - R}} = \lambda \frac{\alpha_1 + \alpha_2 - R}{\alpha_1 + \alpha_2} = \frac{\lambda}{\lambda} = 1.$$

The marginal cost of government revenue is  $\lambda$ , which was computed in part (a) under either solution method:

$$\lambda = \frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 - R}.$$

The marginal utility of income is unaffected by the government revenue requirement while the marginal cost of government revenue is increasing in  $R$  (since  $l = \alpha_1 + \alpha_2$  is the total output,  $R$  must be less than  $\alpha_1 + \alpha_2$ .) Thus the marginal efficiency cost of the tax system (as measured by the marginal cost of government revenue less the marginal utility of income for the individual) is increasing in  $R$ . This result makes intuitive sense as the government must raise revenue using distortionary taxes and the efficiency cost of distortionary taxes is convex in the level of the tax. If the government could raise funds with lump sum taxes, the marginal cost of government revenue would be  $\lambda = \alpha$  and the marginal efficiency cost of the tax system would be  $\alpha - \alpha = 0$ .

## 2. Capital Taxation

This question asks you to numerically explore the welfare implications of capital taxation in the Ramsey setup discussed in class. There is a representative agent with preferences

$$u(c, l) = \frac{1}{1 - \sigma} \left[ c^\theta (1 - l)^{1 - \theta} \right]^{1 - \sigma}$$

over consumption  $c$  and labor  $l$ . The discount factor is  $\beta$ . Technology is Cobb-Douglas with

$$F(k, l) = Ak^\alpha l^{1 - \alpha},$$

and capital  $k$  depreciates at a rate  $\delta$ . Perform your calculations for  $\sigma = 2$ ,  $\sigma = 3$ , and  $\sigma = 4$ .

**(a) Consider a steady state competitive equilibrium of this economy where the tax on capital is  $\kappa = 0$  and there is also no tax on labor income. What is the steady state gross interest rate  $R$ ? Using the steady state resource constraint and the first order conditions from the consumer's utility maximization problem and the firm's profit maximization problem, find 4 equations that implicitly determine the steady state wage  $w$ , consumption  $c$ , labor supply  $l$ , and capital stock  $k$  as a function of the parameters. Calibrate the economy by choosing parameter values for  $\theta$ ,  $A$ ,  $\alpha$ ,  $\delta$ , and  $\beta$  that imply empirically reasonable values for the endogenous variables in the steady state (refer to Cooley (1995), chapter 1). Solve your equations for  $c$ ,  $l$ ,  $k$ , and  $w$ .**

The consumer problem formulated using the present value budget constraint and without labor taxes, is

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \text{ s.t. } \sum_{t=0}^{\infty} q_t (c_t - w_t l_t - T_t) \leq R_0 k_0$$

where  $q_t = \frac{1}{\prod_{i=1}^t R_i}$ .

This leads to first order conditions for  $c_t$ ,  $c_{t+1}$ , and  $l_t$

$$\begin{aligned} \beta^t u_c(c_t, l_t) &= \lambda q_t \\ \beta^{t+1} u_c(c_{t+1}, l_{t+1}) &= \lambda q_{t+1} \\ \beta^t u_l(c_t, l_t) &= -\lambda q_t w_t. \end{aligned}$$

Combining these and taking advantage of the definition of  $q_t$  we get the usual intratemporal and intertemporal optimality conditions

$$\begin{aligned} u_c(c_t, l_t) &= \beta R_{t+1} u_c(c_{t+1}, l_{t+1}) \\ w_t u_c(c_t, l_t) &= -u_l(c_t, l_t). \end{aligned}$$

We assume the existence of a steady state. In the steady state, consumption and labor are constant, thus implying  $R = 1/\beta$ . The gross interest rate is  $R = 1 + (1 - \kappa)(r - \delta)$ , which implies

$$r = \frac{R - 1}{1 - \kappa} + \delta.$$

We assume the presence of profit maximizing firms with the specified technology, which implies

$$\begin{aligned} r &= F_k(k, l) = \alpha A k^{\alpha-1} l^{1-\alpha} \\ w &= F_l(k, l) = (1 - \alpha) A k^{\alpha} l^{-\alpha}. \end{aligned}$$

Solving for the capital labor ratio using the rental rate,

$$\frac{k}{l} = \left( \frac{r}{\alpha A} \right)^{\frac{1}{\alpha-1}}.$$

The resource constraint in steady state is  $c + k = AF(k, l) + (1 - \delta)k$ , which then implies

$$\frac{c}{l} = AF\left(\frac{k}{l}, 1\right) - \delta \frac{k}{l} = A \left(\frac{k}{l}\right)^{\alpha} - \delta \left(\frac{k}{l}\right).$$

Differentiating the given utility function, the marginal utilities with respect to consumption and labor are

$$\begin{aligned} u_c(c, l) &= \left[ c^{\theta} (1 - l)^{1-\theta} \right]^{-\sigma} \theta c^{\theta-1} (1 - l)^{1-\theta} \\ u_l(c, l) &= - \left[ c^{\theta} (1 - l)^{1-\theta} \right]^{-\sigma} c^{\theta} (1 - \theta) (1 - l)^{-\theta}. \end{aligned}$$

Thus the intratemporal optimality condition is

$$\begin{aligned} w \left[ c^{\theta} (1 - l)^{1-\theta} \right]^{-\sigma} \theta c^{\theta-1} (1 - l)^{1-\theta} &= \left[ c^{\theta} (1 - l)^{1-\theta} \right]^{-\sigma} c^{\theta} (1 - \theta) (1 - l)^{-\theta} \\ \Rightarrow w \theta c^{\theta-1} (1 - l)^{1-\theta} &= c^{\theta} (1 - \theta) (1 - l)^{-\theta} \\ \Rightarrow w \theta (1 - l) &= c (1 - \theta) \\ \Rightarrow w \theta (1 - l) &= \frac{c}{l} l (1 - \theta) \\ \Rightarrow l &= \frac{\theta w}{(1 - \theta) \frac{c}{l} + \theta w}. \end{aligned}$$

Values for  $k$ ,  $c$ , and  $y$  can then be computed from  $k = \frac{k}{l}l$ ,  $c = \frac{c}{l}l$ , and  $y = F(k, l)$ . We therefore have a triangular system of equations that allows for the computation of all steady state variables of interest as functions of the parameters:

$$\begin{aligned}
R &= \frac{1}{\beta} \\
r &= \frac{R-1}{1-\kappa} + \delta \\
\frac{k}{l} &= \left(\frac{r}{\alpha A}\right)^{\frac{1}{\alpha-1}} \\
w &= A(1-\alpha) \left(\frac{k}{l}\right)^\alpha \\
\frac{c}{l} &= A \left(\frac{k}{l}\right)^\alpha - \delta \left(\frac{k}{l}\right) \\
l &= \frac{\theta w}{(1-\theta)\frac{c}{l} + \theta w} \\
k &= \frac{k}{l}l \\
c &= \frac{c}{l}l \\
y &= Ak^\alpha l^{1-\alpha}.
\end{aligned}$$

We calibrate the model following Cooley. Capital evolution in the steady state implies  $k = I + (1-\delta)k$  or  $\delta = I/k$ . In 2007 (using data from before the recession in the hope that it better approximates a steady state), investment was \$3.8 trillion and capital was \$47.9 trillion. Thus implying  $\delta = 0.079$ .

Next,

$$\begin{aligned}
\beta &= R^{-1} = (1 + (1-\kappa)(r-\delta))^{-1} \\
&= \left(1 + (1-\kappa) \left(\alpha \left(\frac{k}{y}\right)^{-1} - \delta\right)\right)^{-1}.
\end{aligned}$$

We use the approximation of  $\alpha = 0.33$ . Output in 2007 was \$14.1 trillion. We approximate the capital tax rate as 0.25. Thus  $\beta = 0.99$ . Finally, the intratemporal optimization condition (including labor taxes), requires

$$(1-\alpha)(1-\tau_l)\frac{y}{c} = \frac{1-\theta}{\theta} \frac{l}{1-l}.$$

Using Cooley's choice of  $l = 0.31$ , approximating the labor tax rate as  $\tau_l = 0.28$ , and observing that consumption in 2007 was \$9.8 trillion leads to  $\theta = 0.39$ . Finally, we set

$$A = \frac{y}{k^\alpha l^{1-\alpha}} = 8.6.$$

With values for  $\theta$ ,  $A$ ,  $\alpha$ ,  $\delta$ , and  $\beta$  we can now compute values for the endogenous variables in this economy.

**(b) Using the parameters found in (a), compute the steady state for capital taxes  $\kappa \in [-0.5, 0.5]$ , assuming that the tax revenue is returned in a lump sum fashion to consumers. Plot steady state output, capital, consumption, and labor  $y(\kappa)$ ,  $k(\kappa)$ ,  $c(\kappa)$ , and  $l(\kappa)$  as a function of the capital tax  $\kappa$ . Compute the value of the steady state as**

$$\tilde{V}(k(\kappa)) = \frac{1}{(1-\beta)(1-\sigma)} \left[ c(\kappa)^\theta (1-l(\kappa))^{1-\theta} \right]^{1-\sigma}$$

for given  $\kappa$ . We want to evaluate the welfare cost of this capital tax distortion by comparing this to the value that consumers would obtain if the capital tax were set to zero. A naive approach to this would be to compare  $\tilde{V}(k(\kappa))$  with  $\tilde{V}(k(0))$ . Compute  $\lambda(\kappa)$  such that

$$\frac{1}{(1-\beta)(1-\sigma)} \left[ ((1+\lambda(\kappa))c(\kappa))^\theta (1-l(\kappa))^{1-\theta} \right]^{1-\sigma} = \tilde{V}(k(0)).$$

**Plot  $\lambda(\kappa)$ . How does your result depend on  $\sigma$ ? Interpret.**

Matlab code to create the plots. Note that the steady state values of output, capital, consumption, and labor do not depend on the value of  $\sigma$ . This can be seen by comparing the graphs or referencing the equations derived in part (a). This result makes sense because the parameter  $\sigma$  determines the curvature of the utility function. In the steady state consumption and labor supply never change and the capital-labor ratio is pinned down by the discount rate and the capital taxes, so that the curvature of the utility is of no importance.

As can be seen from the plot of  $\lambda(\kappa)$ , the increase in consumption associated with a tax reform that sets  $\kappa = 0$  is strictly increasing in  $\kappa$ . Furthermore, consumption decreases if the initial capital tax is negative. Given our previous exposure to the Chamley-Judd result, we know this finding is incorrect. This problem arises because the comparison ignores the transition from a given capital stock to a particular steady state capital stock. In steady state solutions in which capital is subsidized, consumption is higher than in the  $\kappa = 0$  steady state but the agent would prefer to eat some of the capital stock now and transition to a steady state with less capital.

**(c) The conclusion in (b) ignores the transition from the capital stock  $k(\kappa)$  to  $k(0)$  once the capital tax  $\kappa$  is reduced to zero. The value function accounting for this solves the Bellman equation**

$$V(k) = \max_{c,l,k'} u(c,l) + \beta V(k')$$

**subject to**

$$c + k' = F(k,l) + (1-\delta)k.$$

**Solve numerically for  $V(k)$  and evaluate it at  $k(\kappa)$ . Again, find  $\lambda(\kappa)$  such that**

$$\frac{1}{(1-\beta)(1-\sigma)} \left[ ((1+\lambda(\kappa))c(\kappa))^\theta (1-l(\kappa))^{1-\theta} \right]^{1-\sigma} = V(k(\kappa)).$$

**Plot  $\lambda(\kappa)$ . How does your result depend on  $\sigma$ ? How does it compare to your result in (b)? Interpret.**

Matlab code (thanks to Florian and Greg for their work) to create the plots.

The code numerically solves for the value function  $V(k)$  using value function iteration. The increase in consumption associated with a tax reform that sets  $\kappa = 0$  is now strictly greater than zero and equal to zero only at an initial capital tax of zero, which is the optimal level. While a capital subsidy could increase the steady state capital labor ratio, and therefore the wage and consumption, the agent prefers to eat the capital now. Furthermore, the plot of  $\lambda(\kappa)$  now depends on the parameter  $\sigma$ . Higher values of  $\sigma$  correspond to greater curvature in the period utility function, and indicate that the agent places greater value on a smooth consumption trajectory. Therefore, for a given initial suboptimal capital stock a higher value of  $\sigma$  makes the costs of the transition to the optimal capital stock more important. This corresponds to the reduced value of  $\lambda(\kappa)$  as  $\sigma$  increases.

### 3. Pigouvian taxation, pollution and jobs

Please read the paper Shimer (2012) “A Framework for Valuing the Employment Consequences of Environmental Regulation”.

**a) Map the model(s) into an Arrow-Debreu Walrasian economy and then apply the general results on Pigouvian taxation that we discussed in class**

Let us map the Shimer model into our notation from class:

- The vector of goods consumed by individual  $i$ :  $x_i = (c_i, d_i, l) = (c_i, d_i, 1)$
- The externality/total/average consumption of dirty goods  $\bar{x} = \int d_i di$
- consumer prices  $q = (p_c, p_d)$
- producer prices  $p = (\frac{p_c}{1+\tau_c}, \frac{p_d}{1+\tau_d})$
- the utility function of individual  $i$ :  $u_i = V(u(c_i, d_i), D)$

Assume that the externality is large enough that optimality requires a reduction in pollution (reallocation).

Now use the formula at the bottom of page 2 of Ivan's lecture notes on corrective taxation to find the optimal Pigouvian tax and use that we are in the equilibrium with reallocation (which implies  $\frac{p_c/q_c}{p_d/q_d} = \frac{1+\tau_d}{1+\tau_c}$ ):

$$\frac{p_c/q_c}{p_d/q_d} = \frac{1 + \tau_d}{1 + \tau_c} = \frac{1}{1 + \int \frac{V_D}{V_{d_i}} di}$$

**b) Compare the results from a) with the results in Shimer (2012) and discuss.**

Let us define  $-\int \frac{V_D}{V_{d_i}} di = \sigma^{d,D}$  as the (average) MRS between dirty goods and pollution and then we get

$$\frac{1 + \tau_d}{1 + \tau_c} = \frac{1}{1 - \sigma^{d,D}}$$

this can be rewritten as equation (2) on p.15 of Shimer:

$$\tau_d - \tau_c = \sigma^{d,D} (1 + \tau_d)$$

Interestingly, the optimal tax does not depend on the cost of reallocation.

**c) Compare the results from a) with the current EPA practices for valuing job losses of regulation and discuss.**

The EPA scores regulation drafts on the basis of expected unemployment costs. According to this model, this is not optimal since only the MRS between dirty goods and pollution should matter.

## 4. Short questions

Are the following statements true, false or uncertain? Please explain your answer.

- **The Pigouvian principle does not apply with heterogeneous agents.**

False. The Pigouvian principle still applies (see Lecture Notes on Corrective Taxation - Pigouvian Taxation with Many Agents) if we have lump-sum taxes varying across individuals. The Pigouvian principle does apply but is modified as a function of the assumptions on how agents contribute to the externality and what their demand structure is. In class, we covered 3 cases. First, if everyone contributes in the same way to aggregate consumption externality, then the surcharge should be based on the weighted sum across agents of the marginal utility of a households with respect to average consumption. Second, if preferences are additively separable and if the agent's consumption impacts the additive externality in a household-specific manner, then the surcharge should equal a weighed average of externalities, weighted by consumer demand derivatives with weights lying in the unit interval. Third, if utility is not separable then the surcharge should equal a weighted average of externalities, weighted by consumer demand derivatives with weights which could lie outside of the unit interval.

- **The fundamental problem of distortive taxation is that lump-sum taxes are unavailable.**

False because (i) some lump-sum taxes exist in the real world (e.g. negative lump-sum intercept in the schedule due to income-tax deductions or transfers from welfare programs) and (ii) the unavailability of a rich enough set of instruments of lump-sum taxes is driven by more fundamental causes (e.g. technological constraints, information problems) and is thus a syndrom rather than the deep, fundamental cause.

If lump sum taxes were feasible in the Ramsey model, then the first best allocation would be attainable. The second-best problem then chooses the right mix of distortive taxes to maximize the representative agent's utility subject to the government's budget constraint. However, the Ramsey framework does not explicitly capture the reasons for ruling out lump-sum taxation. Distributional concerns are a natural reason to resort to distortionary taxation (Mirrlees, 1971). If workers are heterogeneous with respect to their labor productivity, and if this trait is not observable, or if for some reason taxes cannot be conditioned upon them, then society cannot attain almost any of the first-best allocations. By taxing observable differences such as income, redistribution is possible, albeit at a loss in efficiency. Such a trade-off between redistribution and efficiency provides a microfoundation for the role of distortionary taxation. Then the "fundamental problem" can be seen either as an information problem (labor productivity is not observable), or a lack of instruments problem (lump sum taxation conditional say on DNA is not allowed).

Source: *Werning, Tax Smoothing with Redistribution (page 2)*

- **Suppose preferences over consumption and leisure are Cobb-Douglas. Then a proportional tax on labor, used to finance government consumption, does not affect labor supply. Thus taxation in this case is non-distortionary.**

False. To evaluate distortions we need to look at the substitution effect and not at the total effect. The consumption-leisure margin is distorted because the relative price changes here even if total effect on labor supply is zero. The shortest way to see this is by noticing the wedge between the MRS and the MRT:

$$\frac{MRS}{MRT} = 1 + \tau$$

An alternative way to see this is to compute the impact on utility under a lump-sum rebate of the tax (in GE). If the consumer disposes of time endowment  $T$ , then full income equals  $I = PT$  where the price of leisure is  $P = w(1 - t)$  and where leisure is  $l$  and labor  $L$ . Under Cobb-Douglas, utility is given by:  $u = c^{1-\alpha}l^\alpha + \beta G$ . From the following demands, it is obvious that leisure does not change and hence the uncompensated labour supply elasticity is zero

$$c = (1 - \alpha)I$$

$$l = \frac{\alpha I}{P} = \frac{\alpha(PT)}{P} = \alpha T$$

But we can compute the indirect utility function by plugging in consumption and leisure into the utility and show that the compensated labor supply elasticity  $\frac{\partial L^h P}{\partial P L} = \frac{l}{L}$  is positive. Hence if we (i) fix  $G$ , (ii) compensate any variation in  $t$  by matching a higher lump-sum distribution  $R$ , then (i) A higher  $t$  increases  $R$  in which case  $\frac{dL}{dt} < 0$  because of the income effect of  $R$ /leisure is a normal good, (ii) welfare drops  $dV/dt < 0$ . Any increase in the wage tax matched by a lump-sum rebate would decrease utility: the wage tax is distorting. To conclude, there are 3 effects:

1. PE: a higher  $t$  has a positive income effect on labor
2. PE: a lower  $t$  has a negative income effect on labor
3. GE: a higher  $R$  has a negative income effect on labor

In PE, with Cobb-Douglas preferences, effects 1 and 2 cancel each other out. In GE, there is a net negative effect on labor and hence the wage tax is distorting when the tax revenues are redistributed.

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