

14.472 Problem Set 2 Suggested Solutions

Question 1: Because each individual's discount rate is equal to the interest rate (0), there are no savings and borrowing constraints, and the utility of consumption function is the same in every period, the individual will consume the same amount in each period. $x=c$. (I didn't write the i superscript in the answers to clean up notation.)

$$\text{Max } \int_0^R (\log(x) - a) dz + \int_R^T \log(c) dz$$

$$\text{S.T. } \int_0^R (x - w) dz + \int_R^T c dz \leq I$$

$$\text{FOC: } x = c = \frac{1}{\lambda}$$

$$\text{Solution: } \begin{aligned} x &= \frac{w}{a} \\ R &= \frac{T}{a} - \frac{I}{w} \end{aligned}$$

DON'T FORGET CORNER SOLUTION POSSIBILITIES!

$$\text{if } 0 < \frac{T}{a} - \frac{I}{w} < T \text{ then } x = c = \frac{w}{a} \quad R = \frac{T}{a} - \frac{I}{w}$$

$$\text{if } 0 \geq \frac{T}{a} - \frac{I}{w} \text{ then } x = c = \frac{I}{T} \quad R = 0$$

$$\text{if } T \leq \frac{T}{a} - \frac{I}{w} \text{ then } x = c = w + \frac{I}{T} \quad R = T$$

Marginal utility of consumption: a/w . Since $a^L > a^H$ and $w^L < w^H$, then the marginal utility for the low type is higher than the higher type.

2) Same maximization problem, although the budget constraints changes to:

$$\int_0^R (x - (1 - \tau)w - \beta) dz + \int_R^T (c - \beta) dz \leq I$$

The FOC are the same, with the value of λ changing because of the change in the BC.

$$x = \frac{w(1-\tau)}{a}$$

Solution:

$$R = \frac{T}{a} - \frac{I+T\beta}{w(1-\tau)}$$

AGAIN: DON'T FORGET CORNER SOLUTION POSSIBILITIES:

$$\begin{aligned}
 \text{if } 0 < \frac{T}{a} - \frac{I+T\beta}{w(1-\tau)} < T & \text{ then } x = c = \frac{w(1-\tau)}{a} & R = \frac{T}{a} - \frac{I+T\beta}{w(1-\tau)} \\
 \text{if } 0 \geq \frac{T}{a} - \frac{I+T\beta}{w(1-\tau)} & \text{ then } x = c = \frac{I+T\beta}{T} & R = 0 \\
 \text{if } T \leq \frac{T}{a} - \frac{I+T\beta}{w(1-\tau)} & \text{ then } x = c = w(1-\tau) + \beta + \frac{I}{T} & R = T
 \end{aligned}$$

Marginal utility of consumption: $a/(1-\lambda)w$ when an interior solution. The marginal utility of the low type is higher than of the low type, but there is less of a difference between them than in Q1.

The govt budget constraint:

$$\tau(R^L w^L + R^H w^H) \geq \beta(T^L + T^H)$$

3) Same maximization problem with the budget constraint changed to:

$$\int_0^R (x - (1-t(1-\alpha))w - \beta) dz + \int_R^T cz \leq I$$

FOC the same.

$$x = \frac{w(1-t(1-\alpha)) + b}{a}$$

Solution:

$$R = \frac{T}{a} - \frac{I}{b + w(1-t(1-\alpha))}$$

CORNER SOLUTIONS AGAIN POSSIBLE:

$$\begin{aligned}
 \text{if } 0 < \frac{T}{a} - \frac{I}{b + w(1-t(1-\alpha))} < T & \text{ then } x = c = \frac{w(1-t(1-\alpha)) + b}{a} & R = \frac{T}{a} - \frac{I}{b + w(1-t(1-\alpha))} \\
 \text{if } 0 \geq \frac{T}{a} - \frac{I}{b + w(1-t(1-\alpha))} & \text{ then } x = c = \frac{I}{T} & R = 0 \\
 \text{if } T \leq \frac{T}{a} - \frac{I}{b + w(1-t(1-\alpha))} & \text{ then } x = c = \frac{I}{T} + b + w(1-t(1-\alpha)) & R = T
 \end{aligned}$$

the marginal utility of consumption: $a/(b+w(1-t(1-\alpha)))$ with an interior solution. They are the closest in this case (compared to Q1 and Q2).

S.S. Budget constraint:

$$(1-\alpha)t(R^L w^L + R^H w^H) \geq b(R^L + R^H)$$

4) The individual is still facing an earnings tax, now $(1-\alpha)t$ instead of τ , but this does not substantially alter incentives. The main difference is that the flow benefit b is no only available when the individual is working. This provides incentives to delay retirement in Q3 compared to Q2 because:

$$\frac{\partial R}{\partial \beta} < 0$$

$$\frac{\partial R}{\partial b} > 0$$

If accounts had to be annuitized and the annuities were individually fair, this would not change incentives since it would only affect the timing of benefits, not the amount of the benefit. Since there are no liquidity or borrowing constraints, the optimum is not changed.

If accounts had to be annuitized and the annuities were fair for the cohort, the each individual receives a flow benefit ϕ such that $\phi(T^H - R^H + T^L - R^L) = R^L(b + \alpha t w^L) + R^H(b + \alpha t w^H)$. In this set up, there is redistribution from the individual with the larger T-R. When the taxes are paid into a common pot, this changes the budget constraints and alters incentives. Since individuals don't take into account that their retirement decisions will affect the benefit level they receive when they retire, there are now incentives to retire earlier.

5) No savings, no redistribution program, log utility \rightarrow must work entire life.

$$R = T$$

$$x = w$$

$$6). \text{ Max } R \log(w(1 - \tau) + \beta) + (T - R) \log(\beta) - Ra$$

solution:

$$R = \begin{cases} 0 & \text{if } \log(w(1 - \tau) + \beta) - a - \log(\beta) < 0 \\ \in [0, T] & \text{if } \log(w(1 - \tau) + \beta) - a - \log(\beta) = 0 \\ T & \text{if } \log(w(1 - \tau) + \beta) - a - \log(\beta) > 0 \end{cases}$$

MU of consumption is now:

$$\frac{1}{(1 - \tau)w + \beta} \quad \text{if } [0, R]$$

$$\frac{1}{\beta} \quad \text{if } [R, T]$$

The marginal utility of type H while working is less than the marginal utility of type L while working, and both are less than the marginal utility of being retired.

The government's budget constraint: (the same as it was in 2)

$$\tau(R^L w^L + R^H w^H) \geq \beta(T^L + T^H)$$

We cannot have both the high and the low type not working because that would yield no revenue, thus no benefits. You can have equilibria of the low type never working, or working then retiring, or work his entire life. The high type can work then retire, or work entire life.

7) The budget constraint becomes:

$$x = w(1 - t)$$

$$c = \frac{R(\alpha t w + b)}{T - R}$$

same maximization problem. The FOC equates the marginal utility of working with the marginal utility of retiring:

$$\log(w(1 - t)) - a = \log\left(\frac{R(\alpha t w + b)}{T - R}\right) + (T - R)\left(\frac{1}{R} + \frac{1}{T - R}\right)$$

Since this is decreasing in R, and diverges to positive infinity as R goes to 0, and negative infinity as R goes to T, there is a unique R optimum that is given by this equation R*).

The SS budget constraint: (the same as in 3)

$$(1 - \alpha)t(R^L w^L + R^H w^H) \geq b(R^L + R^H)$$

8) In 7, individuals realize that their future benefit payments depend on when they retire (as opposed to just being constraints in the budget constraint). Because In 7 you only receive b when working (as opposed to receiving β anytime in 6), there is a marginal incentive to work in 7. But the actual retirement age could change dramatically depending on the tax system in 6 (corner solutions).

Since the savings/spending decisions are irrational before retirement, we cannot say that the incentive effects of replace the tax in 6 with the tax scheme in 7 is the same, even if the financial effect is the same.

9) Type 1 maximization:

$$\max \int_0^R (\log(1 - t)w - a) dz + \int_R^T \log\left(\frac{R(\alpha t w + b)}{T - R}\right) dz$$

$$ST \ R \geq E$$

Clearly never optimal to retire before E and get no income. So we get a similar result as in 7.

$$R^1 = \begin{cases} E & \text{if } \log(w(1-t)) - a - \log(\alpha atw + b) - \log\left(\frac{E}{T-E}\right) + \frac{T}{E} < 0 \\ R^* & \text{if } \log(w(1-t)) - a - \log(\alpha atw + b) - \log\left(\frac{R}{T-R}\right) + \frac{T}{R} = 0 \end{cases}$$

Since type 2 doesn't make any decision on retirement other than rational savings/consumption after retirement, you get the same result as in 7 for type 2.

$$x = w(1-t)$$

$$c = \frac{E(\alpha tw + b)}{T-E}$$

Thus the SS Budget constraint is the same as in 7.

$$(1-a)t(R^1 w^1 + E w^2) \geq b(R^1 + E)$$

E drops out of the equation if the wages are equal.

Determining the optimal E requires a SWF. Let's assume that SWF includes a maximization of lifetime utilities, even with the irrationality before retirement (assume there is some missing market).

In the case where $R^1 = E$, then decreasing E will increase the utility of the first type since he can now retire earlier. If $a^1 < a^2$, then decreasing E will also increase the utility of the second type, since they would rationally set $R^2 < R^1 = E$, but cannot due to liquidity constraints.

If $R^1 = R^*$, then you'd want to chose E to optimize the utility of the second type. (E doesn't effect the utility of the first type at all in this case)

If $a^1 > a^2$, then the rational type 2 workers would retire later than type 1 workers, it is no longer optimal to set $E = R^{2*}$, since this would reduce the utility of the type 1 worker (who is constrained so $R^1 > E$). The optimal would be somewhere between R^{1*} and R^{2*} , depending on the SWF.