A…Samuelson FOC

<u>Notation</u>

x	consumption
y	labor supply
z	earnings
w	wage
g	public good
$u\left[x,y,g\right]$	utility
f_i	fraction of population type i
p	price of public good

Social welfare maximization:

$$\begin{aligned} &\text{Maximize}_{x,y,g} \quad \sum f_i u^i\left[x_i, y_i, g\right] \\ &\text{subject to:} \quad E + pg + \sum f_i\left(x_i - w_i y_i\right) \le 0 \end{aligned} \tag{1}$$

FOC:

$$f_i u_x^i\left[x_i, y_i, g\right] = \lambda f_i \tag{2}$$

$$f_i u_y^i\left[x_i, y_i, g\right] = \lambda f_i w_i \tag{3}$$

$$\sum f_i u_g^i\left[x_i, y_i, g\right] = \lambda p \tag{4}$$

Implying:

$$\sum f_i \frac{u_g^i\left[x_i, y_i, g\right]}{u_x^i\left[x_i, y_i, g\right]} = p \tag{5}$$

$$\sum f_i w_i \frac{u_g^i\left[x_i, y_i, g\right]}{u_y^i\left[x_i, y_i, g\right]} = p \tag{6}$$

B…Second-Best FOC

R. Boadway and M. Keen, "Public Goods, Self-Selection and Optimal Income Taxation," *International Economic Review* v34, n3 (August 1993), 463-78.

$$\begin{aligned} &\text{Maximize}_{x,y,g} \quad \sum f_i u^i\left[x_i, y_i, g\right] \\ &\text{subject to:} \quad E + pg + \sum f_i\left(x_i - w_i y_i\right) \le 0 \\ &\qquad u^i\left[x_i, y_i, g\right] \ge u^i\left[x_j, y_j w_j / w_i, g\right] \ \text{ for all } i \text{ and } j \end{aligned} \tag{7}$$

Assume two types. Assume the only binding moral hazard constraint is type 1 considering imitating type 2.

$$\begin{aligned} &\text{Maximize}_{x,y,g} \quad f_1 u^1\left[x_1, y_1, g\right] + f_2 u^2\left[x_2, y_2, g\right] \\ &\text{subject to:} \quad E + pg + \sum f_i\left(x_i - w_i y_i\right) \le 0 \\ &\qquad u^1\left[x_1, y_1, g\right] \ge u^1\left[x_2, y_2 w_2 / w_1, g\right] \end{aligned} \tag{8}$$

FOC:

$$f_1 u_x^1\left[x_1, y_1, g\right] - \lambda f_1 + \mu u_x^1\left[x_1, y_1, g\right] = 0 \tag{9}$$

$$f_1 u_y^1\left[x_1, y_1, g\right] - \lambda f_1 w_1 + \mu u_y^1\left[x_1, y_1, g\right] = 0 \tag{10}$$

$$f_2 u_x^2\left[x_2, y_2, g\right] - \lambda f_2 - \mu u_x^1\left[x_2, y_2 w_2 / w_1, g\right] = 0 \tag{11}$$

$$f_2 u_y^2\left[x_2, y_2, g\right] - \lambda f_2 w_2 - \mu u_y^1\left[x_2, y_2 w_2 / w_1, g\right] = 0 \tag{12}$$

$$f_1 u_g^1\left[x_1, y_1, g\right] + f_2 u_g^2\left[x_2, y_2, g\right] + \mu\left(u_g^1\left[x_1, y_1, g\right] - u_g^1\left[x_2, y_2 w_2 / w_1, g\right]\right) = \lambda p \tag{13}$$

Rearranging terms and adding and subtracting $\mu u_x^1\left[x_2, y_2 w_2/w_1, g\right]\left(\frac{u_g^2\left[x_2, y_2, g\right]}{u_x^2\left[x_2, y_2, g\right]}\right)$ to the last equation:

$$
\begin{aligned}
&\left(f_1+\mu\right) u_g^1\left[x_1, y_1, g\right]+\left(f_2-\mu \frac{u_x^1\left[x_2, y_2 w_2/w_1, g\right]}{u_x^2\left[x_2, y_2, g\right]}\right) u_g^2\left[x_2, y_2, g\right] \\
&-\mu\left(u_g^1\left[x_2, y_2 w_2/w_1, g\right]-u_x^1\left[x_2, y_2 w_2/w_1, g\right]\left(\frac{u_g^2\left[x_2, y_2, g\right]}{u_x^2\left[x_2, y_2, g\right]}\right)\right)=\lambda p
\end{aligned}
\tag{14}
$$

Rearranging terms again:

$$
\begin{aligned}
&\left(f_1+\mu\right) u_x^1\left[x_1, y_1, g\right] \frac{u_g^1\left[x_1, y_1, g\right]}{u_x^1\left[x_1, y_1, g\right]}+\left(f_2 u_x^2\left[x_2, y_2, g\right]-\mu u_x^1\left[x_2, y_2 w_2/w_1, g\right]\right) \frac{u_g^2\left[x_2, y_2, g\right]}{u_x^2\left[x_2, y_2, g\right]} \\
&-\mu u_x^1\left[x_2, y_2 w_2/w_1, g\right]\left(\frac{u_g^1\left[x_2, y_2 w_2/w_1, g\right]}{u_x^1\left[x_2, y_2 w_2/w_1, g\right]}-\frac{u_g^2\left[x_2, y_2, g\right]}{u_x^2\left[x_2, y_2, g\right]}\right)=\lambda p
\end{aligned}
$$

(15)

Using the other FOC:

$$\lambda f_1 \frac{u_g^1[x_1, y_1, g]}{u_x^1[x_1, y_1, g]} + \lambda f_2 \frac{u_g^2[x_2, y_2, g]}{u_x^2[x_2, y_2, g]} - \mu u_x^1[x_2, y_2 w_2 / w_1, g]\left(\frac{u_g^1[x_2, y_2 w_2 / w_1, g]}{u_x^1[x_2, y_2 w_2 / w_1, g]} - \frac{u_g^2[x_2, y_2, g]}{u_x^2[x_2, y_2, g]} \right) = \lambda p \tag{16}$$

This reduces to the Samuelson condition if

$$\frac{u_g^1[x_2, y_2 w_2 / w_1, g]}{u_x^1[x_2, y_2 w_2 / w_1, g]} = \frac{u_g^2[x_2, y_2, g]}{u_x^2[x_2, y_2, g]} \tag{17}$$

For example, this holds with everyone having the same utility function that is additive:

$$a[x] + b[y] + c[g]$$

or everyone has a utility function separable in labor and with the same subadditive form:

$$u^i[h[x, g], y].$$

Otherwise, the deviation from the Samuelson condition depends on the MRS for the imitated type compared with that for an imitator, that is, it depends on how the MRS varies with labor supply.

And the conditions for the Samuelson rule to hold are different for different numeraires.