Reference: Eytan Sheshinski, Optimum and Risk-Class Pricing of Annuities, unpublished 2001.

Workers ex ante identical.
no utility discounting
zero interest rate
intertemporally additive preferences
no age variation in preferences
work intensity variable.
All pricing actuarially fair given information

3 period model:
- period 1 – everyone works, representing k identical periods
- period 2 – everyone works
- period 3 – no one works – probabilities π and π' of survival to this period

Between periods 1 and 2 each person may receive a signal that his or her survival probability is π'. Otherwise the survival probability is π. The probability of a signal is $1-\theta$. Note that the arrival of the signal changes only survival probabilities, not preferences.

Notation

V	expected lifetime utility
$u[x_z]$	flow utility of consumption
x_z	consumption in period z before signal time or after no signal
x'_z	consumption in period z after signal
$v[1-y_z]$	flow utility of leisure
y_z	labor supply in period z before signal time or after no signal
y'_z	labor supply in period z after signal
w	wage - the same at all ages

Arrow-Debreu equilibrium assuming symmetric information about survival probabilities

$$\begin{aligned}&\text{Max } k\left(u[x_1]+v[1-y_1]\right)+\theta\left(u[x_2]+v[1-y_2]+\pi u[x_3]\right)+(1-\theta)\left(u[x'_2]+v[1-y'_2]+\pi' u[x'_3]\right)\\ &\text{s. t. } kx_1+\theta x_2+\theta\pi x_3+(1-\theta)x'_2+(1-\theta)\pi' x'_3=w\left(ky_1+\theta y_2+(1-\theta)y'_2\right)\end{aligned} \tag{1}$$

FOC

$$\begin{aligned}&x_1=x_2=x'_2=x_3=x'_3\equiv x\\ &y_1=y_2=y'_2\equiv y\\ &u'[x]=v'[1-y]/w\\ &\left(k+1+\theta\pi+(1-\theta)\pi'\right)x=(k+1)wy\end{aligned} \tag{2}$$

Equilibrium assuming annuitization at the end of period 2 with symmetric information.

$$\begin{aligned}&\text{Max } k\left(u[x_1]+v[1-y_1]\right)+\theta\left(u[x_2]+v[1-y_2]+\pi u[x_3]\right)+(1-\theta)\left(u[x'_2]+v[1-y'_2]+\pi' u[x'_3]\right)\\&\text{s. t. } kx_1+x_2+\pi x_3=w\left(ky_1+y_2\right)\\&\quad kx_1+x'_2+\pi' x'_3=w\left(ky_1+y'_2\right)\end{aligned} \tag{3}$$

FOC

$$\begin{aligned}&u'[x_1]=\lambda+\lambda'=v'[1-y_1]/w\\&\theta u'[x_2]=\theta u'[x_3]=\lambda=\theta v'[1-y_2]/w\\&(1-\theta)u'[x'_2]=(1-\theta)u'[x'_3]=\lambda'=(1-\theta)v'[1-y'_2]/w\end{aligned} \tag{4}$$

Equilibrium assuming annuitization at the end of period 2 with asymmetric information.

$$\begin{aligned}&\text{Max } k\left(u[x_1]+v[1-y_1]\right)+\theta\left(u[x_2]+v[1-y_2]+\pi u[x_3]\right)+(1-\theta)\left(u[x'_2]+v[1-y'_2]+\pi' u[x'_3]\right)\\&\text{s. t. } kx_1+x_2+px_3=w\left(ky_1+y_2\right)\\&\quad kx_1+x'_2+px'_3=w\left(ky_1+y'_2\right)\end{aligned} \tag{5}$$

where the price of an annuity is assumed to break even:

$$p\left(\theta x_3+(1-\theta)x'_3\right)=\pi\theta x_3+\pi'(1-\theta)x'_3 \tag{6}$$

FOC

$$\begin{aligned}&u'[x_1]=\lambda+\lambda'=v'[1-y_1]/w\\&\theta u'[x_2]=\theta\pi u'[x_3]/p=\lambda=\theta v'[1-y_2]/w\\&(1-\theta)u'[x'_2]=(1-\theta)\pi' u'[x'_3]/p=\lambda'=(1-\theta)v'[1-y'_2]/w\end{aligned} \tag{7}$$

To simplify, let us eliminate the first period (k=0).
To see the differences assume that both utility functions are log.
We label V in the 3 cases 1, 2s, and 2p for 1st-best, 2nd-best separating, and 2nd-best pooling.

Arrow-Debreu

$$
\begin{aligned}
&x_2 = x'_2 = x_3 = x'_3 \equiv x \\
&y_2 = y'_2 \equiv y \\
&x = w(1-y) \\
&\left(1+\theta\pi+(1-\theta)\pi'\right)x = wy = (w-x) \\
&x = w/\left(2+\theta\pi+(1-\theta)\pi'\right) \\
&V^1 = \left(1+\theta\pi+(1-\theta)\pi'\right)\log[x]+\log[1-y] \\
&V^1 = \left(2+\theta\pi+(1-\theta)\pi'\right)\left(\log[x]\right)-\left(\log[w]\right) \\
&V^1 = \left(1+\theta\pi+(1-\theta)\pi'\right)\left(\log[w]\right)-\left(2+\theta\pi+(1-\theta)\pi'\right)\left(\log\left[2+\theta\pi+(1-\theta)\pi'\right]\right)
\end{aligned}
\tag{8}
$$

Delayed annuitization with symmetric information

$$\begin{aligned}
&\theta / x_2 = \theta / x_3 = \lambda = \theta / w(1 - y_2) \\
&(1-\theta)/x'_2 = (1-\theta)/x'_3 = \lambda' = (1-\theta)/w(1-y'_2) \\
&x_2 + \pi x_3 = w(y_2) = w - x_2 \\
&x'_2 + \pi' x'_3 = w(y'_2) = w - x'_2
\end{aligned} \tag{9}$$

Solving, we have

$$\begin{aligned}
&(2+\pi)x_2 = w \\
&(2+\pi')x'_2 = w \\
&V^{2s} = \theta\left((2+\pi)\log[x_2] - \log[w]\right) + (1-\theta)\left((2+\pi')\log[x'_2] - \log[w]\right) \\
&V^{2s} = \theta(2+\pi)\log[x_2] + (1-\theta)(2+\pi')\log[x'_2] - \log[w] \\
&V^{2s} = \left(1 + \theta\pi + (1-\theta)\pi'\right)\log[w] - \theta(2+\pi)\log[2+\pi] - (1-\theta)(2+\pi')\log[2+\pi']
\end{aligned} \tag{10}$$

Delayed annuitization with asymmetric information

$$\begin{aligned}
&\theta / x_2 = \theta\pi / p x_3 = \lambda = \theta / w(1 - y_2) \\
&(1-\theta) / x'_2 = (1-\theta)\pi' / p x'_3 = \lambda' = (1-\theta) / w(1 - y'_2) \\
&x_2 + p x_3 = w(y_2) = w - x_2 \\
&x'_2 + p x'_3 = w(y'_2) = w - x'_2
\end{aligned} \tag{11}$$

$$p\left(\theta x_3 + (1-\theta) x'_3\right) = \pi\theta x_3 + \pi'(1-\theta) x'_3 \tag{12}$$

Solving, we have

$$\begin{aligned}
&1/x_2 = \pi / x_3 p \\
&1/x'_2 = \pi' / x'_3\, p \\
&(2+\pi) x_2 = w \\
&(2+\pi') x'_2 = w \\
&V^{2p} = \theta\left(2\log[x_2] + \pi\log[x_3]\right) + (1-\theta)\left(2\log[x'_2] + \pi'\log[x'_3]\right) - \log[w] \\
&V^{2p} = \theta\left((2+\pi)\log[x_2] + \pi\log[\pi/p]\right) + (1-\theta)\left((2+\pi')\log[x'_2] + \pi'\log[\pi'/p]\right) - \log[w] \\
&V^{2p} = \left(1 + \theta\pi + (1-\theta)\pi'\right)\log[w] - \left(\theta\pi + (1-\theta)\pi'\right)\log[p] \\
&-\theta\left((2+\pi)\log[2+\pi] - \pi\log[\pi]\right) - (1-\theta)\left((2+\pi')\log[2+\pi'] - \pi'\log[\pi']\right) \\
&p = \frac{(2+\pi')\theta\pi^2 + (2+\pi)(1-\theta)\pi'^2}{(2+\pi')\theta\pi + (2+\pi)(1-\theta)\pi'}
\end{aligned} \tag{13}$$

Contrasting, we have

$$V^1 = \left(1+\theta\pi+(1-\theta)\pi'\right)\left(\log[w]\right) - \left(2+\theta\pi+(1-\theta)\pi'\right)\left(\log\left[2+\theta\pi+(1-\theta)\pi'\right]\right) \quad (14)$$

$$V^{2s} = \left(1+\theta\pi+(1-\theta)\pi'\right)\log[w] - \theta(2+\pi)\log[2+\pi] - (1-\theta)(2+\pi')\log[2+\pi'] \quad (15)$$

$$\begin{aligned} V^{2p} &= \left(1+\theta\pi+(1-\theta)\pi'\right)\log[w] - \left(\theta\pi+(1-\theta)\pi'\right)\log[p] \\ &-\theta\left((2+\pi)\log[2+\pi]-\pi\log[\pi]\right)-(1-\theta)\left((2+\pi')\log[2+\pi']-\pi'\log[\pi']\right) \\ p &= \frac{(2+\pi')\theta\pi^2+(2+\pi)(1-\theta)\pi'^2}{(2+\pi')\theta\pi+(2+\pi)(1-\theta)\pi'} \end{aligned} \quad (16)$$

Subtracting, we have

$$\begin{aligned} V^{2p}-V^{2s} &= \theta\pi\log[\pi]+(1-\theta)\pi'\log[\pi']-\left(\theta\pi+(1-\theta)\pi'\right)\log[p] \\ p &= \frac{(2+\pi')\theta\pi^2+(2+\pi)(1-\theta)\pi'^2}{(2+\pi')\theta\pi+(2+\pi)(1-\theta)\pi'} \end{aligned} \quad (17)$$