Notation

x	consumption
y	labor supply
z	earnings
w	wage
g_h	public good contribution by household h
g	public good contribution by government
G	total of public good contributions
$u\left[x, y, G\right]$	utility
$v\left[g\right]$	warm glow utility
f_i	fraction of population type i
p	price of public good
N	Number of households

$G = g + \sum g_i$

Assume two types. Assume the only binding moral hazard constraint is type 1 considering imitating type 2.

Standard problem with government provision: Model 1. This is Boadway and Keen.

$$\begin{aligned} &\text{Maximize}_{x,y,g} && f_1 u^1\left[x_1, y_1, G\right] + f_2 u^2\left[x_2, y_2, G\right] \\ &\text{subject to:} && E + pG + N\sum f_i\left(x_i - w_i y_i\right) \le 0 \\ &&& u^1\left[x_1, y_1, G\right] \ge u^1\left[x_2, y_2 w_2 / w_1, G\right] \end{aligned} \quad (1)$$

If G enters additively, we would not expect to find a private contribution at the optimum even if one were allowed since $\sum MRS$ equal to MRT implies that any single MRS is less than the MRT.

Now allow subsidized donations: Model 2. By a combination of subsidizing donations and direct government grants, the government still controls the level of public goods. We assume full nonlinearity in taxes:

$$\begin{aligned} &\text{Maximize}_{x,y,g} && f_1 u^1\left[x_1, y_1, G\right] + f_2 u^2\left[x_2, y_2, G\right] \\ &\text{subject to:} && E + pG + N\sum f_i\left(x_i - w_i y_i\right) \le 0 \\ &&& u^1\left[x_1, y_1, G\right] \ge u^1\left[x_2, y_2 w_2 / w_1, G + g_2 - g_1\right] \end{aligned} \quad (2)$$

In order to make the incentive compatibility constraint as weak as possible (and assuming it continues to bind) we want donations by type 2 to be small and donations by type 1 to be large. Thus, with nonnegativity constraints on donations, the optimum has

$$\begin{aligned} &g_2 = 0 \\ &Nf_1 g_1 = G \end{aligned} \tag{3}$$

We would expect the role of donations to change the Samuelson condition. To check that, let us consider the case of additive preferences,

$$a[x] - b[y] + c[g]$$

so that the Samuelson rule holds in Model 1.

$$\begin{aligned} \text{Maximize}_{x,y,g} \quad & f_1\left(a[x_1] - b[y_1] + c[G]\right) + f_2\left(a[x_2] - b[y_2] + c[G]\right) \\ \text{subject to:} \quad & E + pG + N\sum f_i\left(x_i - w_i y_i\right) \le 0 \\ & \left(a[x_1] - b[y_1] + c[G]\right) \ge \left(a[x_2] - b[y_2 w_2 / w_1] + c[G + g_2 - g_1]\right) \\ & 0 \le g_2;\ 0 \le g_1;\ G = Nf_2 g_2 + Nf_1 g_1 \end{aligned} \tag{4}$$

Assuming $\mu \neq 0$, we have the two corner conditions $g_2 = 0, Nf_1 g_1 = G$. Let us substitute and eliminate these variables and their nonnegativity constraints.

$$\begin{aligned} \text{Maximize}_{x,y,g} \quad & f_1\left(a[x_1] - b[y_1] + c[G]\right) + f_2\left(a[x_2] - b[y_2] + c[G]\right) \\ \text{subject to:} \quad & E + pG + N\sum f_i\left(x_i - w_i y_i\right) \le 0 \\ & \left(a[x_1] - b[y_1] + c[G]\right) \ge \left(a[x_2] - b[y_2 w_2 / w_1] + c\left[G\left(1 - \frac{1}{Nf_1}\right)\right]\right) \end{aligned} \tag{5}$$

This has the FOC:

$$f_1 a'[x_1] - \lambda N f_1 + \mu a'[x_1] = 0 \tag{6}$$

$$-f_1 b'[y_1] + \lambda N f_1 - \mu b'[y_1] = 0 \tag{7}$$

$$f_2 a'[x_2] - \lambda N f_2 - \mu a'[x_2] = 0 \tag{8}$$

$$-f_2 b'[y_2] + \lambda N f_2 + \mu b'[y_2 w_2 / w_1] w_2 / w_1 = 0 \tag{9}$$

$$(f_1 + f_2) c'[G] - \lambda p + \mu\left(c'[G] - c'\left[G\left(1 - \frac{1}{Nf_1}\right)\right]\left(1 - \frac{1}{Nf_1}\right)\right) = 0 \tag{10}$$

Doing the same analysis as before:

$$f_1 a'[x_1]\frac{c'[G]}{a'[x_1]} + f_2 a'[x_2]\frac{c'[G]}{a'[x_2]} - \lambda p + \mu\left(c'[G] - c'\left[G\left(1 - \frac{1}{Nf_1}\right)\right]\left(1 - \frac{1}{Nf_1}\right)\right) = 0 \tag{11}$$

$$(\lambda N f_1 - \mu a'[x_1])\frac{c'[G]}{a'[x_1]} + (\lambda N f_2 + \mu a'[x_2])\frac{c'[G]}{a'[x_2]} + \mu\left(c'[G] - c'\left[G\left(1 - \frac{1}{Nf_1}\right)\right]\left(1 - \frac{1}{Nf_1}\right)\right) = \lambda p \tag{12}$$

$$\lambda N f_1 \frac{c'[G]}{a'[x_1]} + \lambda N f_2 \frac{c'[G]}{a'[x_2]} + \mu\left(c'[G] - c'\left[G\left(1 - \frac{1}{Nf_1}\right)\right]\left(1 - \frac{1}{Nf_1}\right)\right) = \lambda p \tag{13}$$

Thus the Samuelson condition may not hold. It depends on the shape of $c'[G]G$. For example we get the Samuelson condition with the log function. What matters is whether more public goods weaken the incentive compatibility constraint at the margin. While more public good means a bigger difference in contributions by type, the evaluation happens at a place where marginal utilities are lower. So it can go either way.

The model changes if we have a warm glow that does or does not enter the SWF:
Warm glow version which does not enter the SWF has the same corner conditions:

$$\begin{aligned}
&\text{Maximize}_{x,y,g} \quad f_1 u^1\left[x_1, y_1, G\right] + f_2 u^2\left[x_2, y_2, G\right] \\
&\text{subject to:} \quad E + pG + N\sum f_i\left(x_i - w_i y_i\right) \le 0 \\
&\qquad u^1\left[x_1, y_1, G\right] + v^1\left[g^1\right] \ge u^1\left[x_2, y_2 w_2 / w_1, G + g^2 - g^1\right] + v^1\left[g^2\right]
\end{aligned} \tag{14}$$

Warm glow version with warm glow in SWF:

$$\begin{aligned}
&\text{Maximize}_{x,y,g} \quad f_1\left(u^1\left[x_1, y_1, G\right] + v^1\left[g^1\right]\right) + f_2\left(u^2\left[x_2, y_2, G\right] + v^2\left[g^2\right]\right) \\
&\text{subject to:} \quad E + pG + N\sum f_i\left(x_i - w_i y_i\right) \le 0 \\
&\qquad u^1\left[x_1, y_1, G\right] + v^1\left[g^1\right] \ge u^1\left[x_2, y_2 w_2 / w_1, G + g^2 - g^1\right] + v^1\left[g^2\right]
\end{aligned} \tag{15}$$

I don't know yet how this comes out.

These correspond to complex taxation. Restrictions on how contributions and earnings interact would give a different structure. For example, see Saez.

Missing here are the costs of fund-raising for private charities. Missing also is the difference in private and public allocation of resources across different charities. Partially this represents different political processes, which should be thought about. Partially it would be captured by a vector of public goods and a restriction on the tax function that taxes must depend on the sum of resources given to charity, not each donation separately.