14.472 Handout on Optimal Tax Treatment of Private Contributions for Public Goods with and without Warm Glow Preferences

February 2004

1 Standard Preferences

1.1 Optimal Public Provision

$$\text{Maximize}_{c,G} \quad \sum N_n \left(u\left[c_n\right] - a_{nn} + b_n v\left[G\right]\right)$$

$$\text{subject to:} \quad E + pG + \sum N_n \left(c_n - n\right) \leq 0$$

$$u\left[c_n\right] - a_{nn} + b_n v\left[G\right] \geq u\left[c_m\right] - a_{mn} + b_n v\left[G\right] \text{ for } m < n \text{ for all } n \tag{1}$$

Forming a Lagrangian, we have

$$L = \sum N_n (u[c_n] - a_{nn} + b_n v[G]) - \lambda \left(E + pG + \sum N_n (c_n - n) \right) \quad (2)$$

$$+ \sum \sum \mu_{nm} (u[c_n] - a_{nn} + b_n v[G] - (u[c_m] - a_{mn} + b_n v[G]))$$

The FOC for public good provision is

$$\sum N_n b_n v'[G] = \lambda p \quad (3)$$

1.2 Subsidized Private Provision

$$c_n = x_n - (1 - s_n)\, p g_n \tag{4}$$

Utility can be written:

$$u\left[x_n - (1 - s_n)\, p g_n\right] - a_{nn} + b_n v\left[G_{\sim n} + g_n\right] \tag{5}$$

The contributions for workers who make positive contributions satisfy

$$(1 - s_n)\, p u'[c_n] = (1 - s_n)\, p u'\left[x_n - (1 - s_n)\, p g_n\right] = b_n v'\left[G_{\sim n} + g_n\right] = b_n v'[G] \tag{6}$$

Social welfare maximization

$$\text{Maximize}_{c,G,g} \quad \sum N_n \left(u[c_n] - a_{nn} + b_n v[G]\right)$$

$$\begin{aligned} \text{subject to:} \quad & E + pG + \sum N_n (c_n - n) \leq 0 \\ & u[c_2] - a_{22} + b_2 v[G] \geq u[c_1] - a_{12} + b_2 v[G - g_2 + g_1] \\ & G \geq \sum N_n g_n;\ g_n \geq 0 \text{ for all } n \end{aligned} \tag{7}$$

The optimum will have one of two forms - either both have the same consumption and (generically) the incentive compatibility constraint is not binding, or the optimum will have the incentive compatibility constraint binding.

Note that the subsidy rate for type 1 can exceed the subsidy rate for type 2 and still support the optimum. Using the maximal subsidy for type 1 that still leaves a zero contribution, we have:

$$(1 + s_1)\, pu'[c_1] = b_1 v'[G] \tag{8}$$

$$(1 + s_2)\, pu'[c_2] = b_2 v'[G] \tag{9}$$

With $b_1 = b_2$, we have $s_1 \geq s_2$ since $c_1 \leq c_2$ with strict inequality if the incentive compatibility constraint is binding.

If the incentive compatibility constraint binds, the constraint becomes

$$u[c_2] - a_{22} + b_2 v[G] = u[c_1] - a_{12} + b_2 v[G - G/N_2] \tag{10}$$

The FOC for the public good satisfies:

$$\sum N_n b_n v'[G] = \lambda p - \mu b_2 (v'[G] - v'[G(1 - 1/N_2)](1 - 1/N_2)) \tag{11}$$

This may or may not satisfy the Samuelson rule and can deviate in either direction depending on the shape of the public good utility function, that is the sign of $(v'[G] - v'[G(1 - 1/N_2)](1 - 1/N_2))$. If v'' is small, this is close to the Samuelson rule.

2 Warm Glow Preferences

With preferences $u[c_n] - a_{nn} + b_n v[G] + w[g_n]$, the FOC for individual donations for workers who make positive contributions satisfies

$$\begin{aligned}(1-s_n)\,pu'[c_n] &= (1-s_n)\,pu'[x_n-(1-s_n)\,pg_n] \\ &= b_n v'[G_{\sim n}+g_n]+w'[g_n]=b_n v'[G]+w'[g_n]\end{aligned} \tag{12}$$

The donation level with a zero subsidy (the minimum donation level), $g[c_n, G]$, satisfies the individual FOC

$$pu'[c_n] = pu'[x_n - pg_n] = b_n v'[G_{\sim n}+g_n]+w'[g_n]=b_n v'[G]+w'[g_n] \tag{13}$$

Differentiating the implicit equation,

$$\frac{\partial g}{\partial c} = \frac{pu''}{w''} > 0, \frac{\partial g}{\partial G} = \frac{-bv''}{w''} < 0. \tag{14}$$

2.1 Social Welfare Optimization

Considering the two-types model with one public good, we can write a formulation allowing warm glow to enter ($\theta = 1$) and not enter ($\theta = 0$) social welfare.

$$\text{Maximize}_{c,G} \quad \sum N_n \left(u\left[c_n\right] - a_{nn} + bv\left[G\right] + \theta w\left[g_n\right]\right)$$

$$\begin{aligned} \text{subject to:} \quad & E + pG + \sum N_n \left(c_n - n\right) \leq 0 \\ & u\left[c_2\right] - a_{22} + bv\left[G\right] + w\left[g_2\right] \geq u\left[c_1\right] - a_{12} + bv\left[G - g_2 + g_1\right] + w\left[g_1\right] \\ & G \geq \sum N_n g_n; \; g_n \geq g\left[c_n, G\right] \text{ for all } n \end{aligned} \tag{15}$$

Differentiating we have lots of FOC:

$$\frac{\partial L}{\partial c_1} = N_1 \left(u'\left[c_1\right] - \lambda\right) - \mu u'\left[c_1\right] - \xi_1 \frac{\partial g\left[c_1, G\right]}{\partial c} = 0 \tag{16}$$

$$\frac{\partial L}{\partial c_2} = N_2 \left(u'\left[c_2\right] - \lambda\right) + \mu u'\left[c_2\right] - \xi_2 \frac{\partial g\left[c_2, G\right]}{\partial c} = 0 \tag{17}$$

$$\frac{\partial L}{\partial G} = \sum \left(N_n bv'\left[G\right] - \xi_n \frac{\partial g\left[c_n, G\right]}{\partial G}\right) - \lambda p + \mu b \left(v'\left[G\right] - v'\left[G - g_2 + g_1\right]\right) + \nu = 0 \tag{18}$$

$$\frac{\partial L}{\partial g_1} = N_1 \theta w'\left[g_1\right] - \mu \left(bv'\left[G - g_2 + g_1\right] + w'\left[g_1\right]\right) - \nu N_1 + \xi_1 = 0 \tag{19}$$

$$\frac{\partial L}{\partial g_2} = N_2 \theta w'\left[g_2\right] + \mu \left(bv'\left[G - g_2 + g_1\right] + w'\left[g_2\right]\right) - \nu N_2 + \xi_2 = 0 \tag{20}$$

We consider the FOC separately for the values of θ of 0 and 1.

2.2 Warm Glow Preferences that do not enter Social Welfare: $\theta = 0$

Increasing the donation of the high type while lowering the donation of the low type weakens the incentive compatibility constraint while having no other effects, until the lower limit on g_1 is hit. Similarly, donations by the high type dominate public provision. Thus we know that $g_1 = g[c_1, G]$. Assuming that public good supply is less than optimal without any subsidization, we also have no binding minimum donation constraint for the high type, $\xi_2 = 0$. Thus we can write the FOC (16) and (17) for the case $\theta = 0$ as

$$\frac{\partial L}{\partial c_1} = N_1 (u'[c_1] - \lambda) - \mu u'[c_1] - \xi_1 \frac{\partial g[c_1, G]}{\partial c} = 0 \tag{21}$$

$$\frac{\partial L}{\partial c_2} = N_2 (u'[c_2] - \lambda) + \mu u'[c_2] = 0 \tag{22}$$

Turning to the other FOC, with g_1 at its minimum, (18) and (20) become

$$\frac{\partial L}{\partial G} = bv'[G] \sum N_n - \xi_1 \frac{\partial g[c_1, G]}{\partial G} - \lambda p + \mu b (v'[G] - v'[G - g_2 + g_1]) + \nu = 0 \tag{23}$$

$$\frac{\partial L}{\partial g_2} = \mu (bv'[G - g_2 + g_1] + w'[g_2]) - \nu N_2 = 0 \tag{24}$$

It is plausible for all but very large donors that v'' is small enough that $v'[G] - v'[G - g_2 + g_1]$ is very small. Then the FOC for public good supply is approximately

$$\frac{\partial L}{\partial G} \approx bv'[G] \sum N_n - \xi_1 \frac{\partial g[c_1, G]}{\partial G} - \lambda p + \nu = 0 \tag{25}$$

We can express the deviation from the Samuelson condition as

$$bv'[G] \sum N_n - \lambda p \approx \xi_1 \frac{\partial g[c_1, G]}{\partial G} - \nu = \xi_1 \frac{\partial g[c_1, G]}{\partial G} - \frac{\mu}{N_2} (bv'[G - g_2 + g_1] + w'[g_2]) \tag{26}$$

Using again the assumption of a small v'', and so approximate constancy of v', and the FOC for individual donations, we can write this as

$$bv'[G] \sum N_n - \lambda p = \xi_1 \frac{\partial g[c_1, G]}{\partial G} - \frac{\mu}{N_2} (1 - s_2) pu'[c_2] \tag{27}$$

From the FOC for consumption, this can be written as

$$bv'[G] \sum N_n - \lambda p \approx \xi_1 \frac{\partial g[c_1, G]}{\partial G} + (1 - s_2) p (u'[c_2] - \lambda) \tag{28}$$

or

$$bv'[G] \sum N_n \approx \xi_1 \frac{\partial g[c_1, G]}{\partial G} + s_2 p \lambda + (1 - s_2) pu'[c_2] \tag{29}$$

2.3 Warm Glow Preferences that do enter Social Welfare: $\theta = 1$

The FOC (16) - (20) become

$$\frac{\partial L}{\partial c_1} = N_1 (u'[c_1] - \lambda) - \mu u'[c_1] - \xi_1 \frac{\partial g[c_1, G]}{\partial c} = 0 \quad (30)$$

$$\frac{\partial L}{\partial c_2} = N_2 (u'[c_2] - \lambda) + \mu u'[c_2] - \xi_2 \frac{\partial g[c_2, G]}{\partial c} = 0. \quad (31)$$

$$\frac{\partial L}{\partial G} = \sum \left(N_n b v'[G] - \xi_n \frac{\partial g[c_n, G]}{\partial G} \right) - \lambda p + \mu b \left(v'[G] - v'[G - g_2 + g_1] \right) + \nu = 0 \quad (32)$$

$$\frac{\partial L}{\partial g_1} = N_1 w'[g_1] - \mu \left(b v'[G - g_2 + g_1] + w'[g_1] \right) - \nu N_1 + \xi_1 = 0 \quad (33)$$

$$\frac{\partial L}{\partial g_2} = N_2 w'[g_2] + \mu \left(b v'[G - g_2 + g_1] + w'[g_2] \right) - \nu N_2 + \xi_2 = 0 \quad (34)$$

Donations of the low type may be subsidized. The dominance of donations of the high type over public provision remains true.

If donations of both types are subsidized ($\xi_1, \xi_2 = 0$), the private consumption FOC have the same form as with fully public provision.

$$\frac{\partial L}{\partial c_1} = N_1 (u'[c_1] - \lambda) - \mu u'[c_1] = 0 \quad (35)$$

$$\frac{\partial L}{\partial c_2} = N_2 (u'[c_2] - \lambda) + \mu u'[c_2] = 0 \quad (36)$$

Assuming that the incentive compatibility constraint is binding ($\mu > 0$), we have

$$\frac{u'[c_1]}{u'[c_2]} = \frac{1 + \mu/N_2}{1 - \mu/N_1} > 1 \tag{37}$$

If donations of both types are subsidized ($\xi_1, \xi_2 = 0$):

$$\frac{\partial L}{\partial g_1} = N_1 w'[g_1] - \mu \left(bv'[G - g_2 + g_1] + w'[g_1]\right) - \nu N_1 = 0 \tag{38}$$

$$\frac{\partial L}{\partial g_2} = N_2 w'[g_2] + \mu \left(bv'[G - g_2 + g_1] + w'[g_2]\right) - \nu N_2 = 0 \tag{39}$$

Solving for $w'[g_n]$ we have

$$\left(1 - \frac{\mu}{N_1}\right) w'[g_1] = \nu + \frac{\mu}{N_1} bv'[G - g_2 + g_1] \tag{40}$$

$$\left(1 + \frac{\mu}{N_2}\right) w'[g_2] = \nu - \frac{\mu}{N_2} bv'[G - g_2 + g_1] \tag{41}$$

The total utility gain from a donation by the lower type is

$$\begin{aligned} bv'[G] + w'[g_1] &= bv'[G] + \left(\nu + \frac{\mu}{N_1}\left(bv'[G - g_2 + g_1]\right)\right) / \left(1 - \frac{\mu}{N_1}\right) \\ &= \left(bv'[G] + \nu + \frac{\mu}{N_1}\left(bv'[G - g_2 + g_1] - bv'[G]\right)\right) / \left(1 - \frac{\mu}{N_1}\right) \end{aligned} \tag{42}$$

We can evaluate the subsidy level.

$$(1 - s_1) = \frac{bv'[G] + w'[g_1]}{pu'[c_1]} = \frac{bv'[G] + \nu + \frac{\mu}{N_1}\left(bv'[G - g_2 + g_1] - bv'[G]\right)}{p\lambda} \tag{43}$$

Similarly, for the high type

$$(1 - s_2) = \frac{bv'[G] + w'[g_2]}{pu'[c_2]} = \frac{bv'[G] + \nu - \frac{\mu}{N_2}\left(bv'[G - g_2 + g_1] - bv'[G]\right)}{p\lambda} \tag{44}$$

Taking the ratio

$$\frac{1 - s_1}{1 - s_2} = \frac{bv'[G] + \nu + \frac{\mu}{N_1}\left(bv'[G - g_2 + g_1] - bv'[G]\right)}{bv'[G] + \nu - \frac{\mu}{N_2}\left(bv'[G - g_2 + g_1] - bv'[G]\right)} > 1 \tag{45}$$

Thus the high type receives a higher subsidy. If, as is plausible, v'' is small relative to individual donations, then $(1 - s_1) / (1 - s_2) \approx 1.$

If donations of both types are subsidized ($\xi_1, \xi_2 = 0$) and v'' is small, the FOC for public good level becomes approximately:

$$bv'[G] \sum N_n - \lambda p + \nu \approx 0 \tag{46}$$

$$\begin{aligned} bv'[G] \sum N_n &\approx \lambda p - \nu = \lambda p - w'[g_2] - \frac{\mu}{N_2}\left(bv'[G] + w'[g_2]\right) \\ &= \lambda p - w'[g_2] - \frac{\mu}{N_2}(1 - s_2)\, pu'[c_2] \\ &= \lambda p - w'[g_2] + (1 - s_2)\, p\left(u'[c_2] - \lambda\right) \\ &= s_2 \lambda p + (1 - s_2)\, pu'[c_2] - w'[g_2] \end{aligned} \tag{47}$$

If w' is much larger than v',then the last two terms approximately cancel and the FOC is as if the only cost of public goods were the marginal cost to the government of the marginal contribution by the high type. Similarly, in terms of the marginal utilities of type 1.

$$bv'[G]\sum N_n \approx s_1\lambda p + (1 - s_1)\, pu'[c_1] - w'[g_1] \tag{48}$$

3 Notation

n	skill index (equal to productivity)
N_n	number of workers of skill n
x_n	compensation at job of skill n
c_n	consumption of worker holding job of skill n
$u[c_n]$	utility of private good consumption
a_{mn}	disutility of labor for a worker of skill n holding a job of skill m
G	public good supply
$b_n v[G]$	utility of public good consumption
g_n	donation of worker holding a job of skill n
$w[g_n]$	warm glow utility
$G_{\sim n}$	public good provision net of donation of a worker holding a job of skill n
p	cost per unit of the public good
E	other public expenditures
s_n	donation subsidy rate for worker in job of skill n
λ	Lagrangian on the resource constraint
μ_n	Lagrangians on the incentive compatibility constraints
ν	Lagrangian on the adding up constraint for public good provision
ξ_n	Lagrangians on the minimum donation constraints
θ	index to distinguish SWF with and without warm glow utility