<u>Notation</u>

x_i	consumption in period i
y	labor supply
z	net of tax earnings
w	wage
$u[x, y]$	utility
f_i	number of workers of type i
p	producer price of good 2
q	consumer price of good 2

There are two theorems to look at, both involving a lack of taxation of commodities in the presence of optimal income taxation. One has general nonlinear taxation, while the other has linear commodity taxation. Both are shown in the two-types model.

A. Full nonlinear taxation

Social welfare maximization assuming full nonlinear taxation:

$$\begin{aligned} &\text{Maximize}_{x,y} \quad \sum f_i u^i\left[x_1^i, x_2^i, y^i\right] \\ &\text{subject to:} \quad E + \sum f_i\left(x_1^i + p x_2^i - w_i y^i\right) \le 0 \\ &\qquad u^i\left[x_1^i, x_2^i, y^i\right] \ge u^i\left[x_1^j, x_2^j, y^j w_j / w_i\right] \text{ for all } i \text{ and } j \end{aligned} \tag{1}$$

Assume that the only binding constraint is type 1 imitating type 2.

Then we have the FOC for the consumption levels (assuming interior solutions)

$$f_1 u^1_{x_1}\left[x^1_1, x^1_2, y^1\right] - \lambda\left(f_1\right) = -\mu\left(u^1_{x_1}\left[x^1_1, x^1_2, y^1\right]\right) \tag{2}$$

$$f_1 u^1_{x_2}\left[x^1_1, x^1_2, y^1\right] - \lambda\left(pf_1\right) = -\mu\left(u^1_{x_2}\left[x^1_1, x^1_2, y^1\right]\right) \tag{3}$$

$$f_2 u^2_{x_1}\left[x^2_1, x^2_2, y^2\right] - \lambda\left(f_2\right) = \mu\left(u^1_{x_1}\left[x^2_1, x^2_2, y^2 w_2 / w_1\right]\right) \tag{4}$$

$$f_2 u^2_{x_2}\left[x^2_1, x^2_2, y^2\right] - \lambda\left(pf_2\right) = \mu\left(u^1_{x_2}\left[x^2_1, x^2_2, y^2 w_2 / w_1\right]\right) \tag{5}$$

Taking ratios, we have

$$\frac{u^1_{x_1}\left[x^1_1, x^1_2, y^1\right]}{u^1_{x_2}\left[x^1_1, x^1_2, y^1\right]} = p \tag{6}$$

$$\frac{f_2 u^2_{x_1}\left[x^2_1, x^2_2, y^2\right] - \mu u^1_{x_1}\left[x^2_1, x^2_2, y^2 w_2 / w_1\right]}{f_2 u^2_{x_2}\left[x^2_1, x^2_2, y^2\right] - \mu u^1_{x_2}\left[x^2_1, x^2_2, y^2 w_2 / w_1\right]} = p \tag{7}$$

The first condition is the usual lack of marginal taxation at the top of the earnings distribution. The second condition involves no taxation if

$$\frac{u^2_{x_1}\left[x^2_1, x^2_2, y^2\right]}{u^2_{x_2}\left[x^2_1, x^2_2, y^2\right]} = \frac{u^1_{x_1}\left[x^2_1, x^2_2, y^2 w_2 / w_1\right]}{u^1_{x_2}\left[x^2_1, x^2_2, y^2 w_2 / w_1\right]} \tag{8}$$

A sufficient condition for the second to give no taxation is separability and the same sub-utility function, $u^i = \tilde{u}^i\left[h\left[x_1, x_2\right], y\right]$, which implies

$$\frac{u^i_{x_1}\left[x_1, x_2, y\right]}{u^i_{x_2}\left[x_1, x_2, y\right]} = \frac{\tilde{u}^i_h\left[h\left[x_1, x_2\right], y\right]h_1\left[x_1, x_2\right]}{\tilde{u}^i_h\left[h\left[x_1, x_2\right], y\right]h_2\left[x_1, x_2\right]} = \frac{h_1\left[x_1, x_2\right]}{h_2\left[x_1, x_2\right]} \quad (9)$$

Note that this has assumed that the corner conditions of nonnegative consumption are not binding.

B. Nonlinear income taxation and linear commodity taxation

The extension to linear consumption taxes follows from this being a more constrained optimum than with full nonlinear taxation. That is, there is an additional constraint on allowable (x, y) vectors. Yet the optimum without this constraint is feasible with the extra constraint.

To proceed with this problem, we could use a mixed direct-indirect utility function:

$$\begin{aligned} v^i\left[y^i, z^i, q\right] = \text{Maximize}_x \quad & u^i\left[x^i_1, x^i_2, y^i\right] \\ \text{subject to:} \quad & x^i_1 + qx^i_2 = z^i \end{aligned} \quad (10)$$