C…Nash equilibrium with private provision
Noncooperative, not cooperative equilibrium

T. Bergstrom, L. Blume, and H. Varian, "On the Private Provision of Public Goods," *Journal of Public Economics* 29 (1986), 25-49.

<u>Notation</u>

x_i	consumption of household i
b_i	income of household i
g_i	public good contribution of household i
G	public good contribution of government
g	public good level: $g = G + \sum g_i$
$u^i\left[x_i, g\right]$	utility of household i - concave
f_i	fraction of population of type i
p	price of public good, set equal to 1
t_i	income tax of household i

Assume two types, with equal numbers.
Private contribution equilibrium (no government activity)

Individual choice, assuming $u_x^i\left[0, g\right] = \infty$

$$\begin{aligned} &\text{Maximize}_{x,g} \quad u^i\left[x_i, g_i + g_j\right] \\ &\text{subject to:} \quad x_i + g_i \le b_i \end{aligned} \tag{1}$$

FOC:

$$u_x^i\left[x_i, g_i + g_j\right] \ge u_g^i\left[x_i, g_i + g_j\right] \tag{2}$$

with equality if $g_i > 0$.

Equilibrium is simultaneous solution of two FOC for two agents.

Four possible equilibria:

A no contributions: $g_1 = g_2 = 0$

B_i contributions only by i: $g_i > 0; g_j = 0$

C contributions by both: $g_i > 0; g_j > 0$

To keep it simple, assume additive utility functions:

$$u^i\left[x_i, g\right] = v_i\left[x_i\right] + w_i\left[g\right] \tag{3}$$

This makes preferences for both goods normal.
Then the equilibrium is:

$$v_1'\left[b_1 - g_1\right] \geq w_1'\left[g_1 + g_2\right] \tag{4}$$

$$v_2'\left[b_2 - g_2\right] \geq w_2'\left[g_1 + g_2\right] \tag{5}$$

D…Comparative statics of government provision with financing

Assume that the government provides G, financed by (nonnegative) taxes on 1 and 2 and possibly an outsider.

$$G \geq t_1 + t_2 \tag{6}$$

With equilibrium of type A, neither taxes nor government provision encourages private contribution:

$$g = G \tag{7}$$

With equilibrium of type B_i, there are two possibilities. Government contributions/taxes might be so large as to have equilibrium of type A, or we still have an equilibrium of type B_i:

$$v_i'\left[b_i - t_i - g_i\right] = w_i'\left[g_i + G\right] \tag{8}$$

Assume that taxes are proportional to government spending, with constant s_i.

$$t_i = s_i G \tag{9}$$

Then differentiating the equilibrium condition, we have:

$$\frac{dg_i}{dG} = -\frac{-s_i v_i'' - w_i''}{-v_i'' - w_i''} = -\frac{-s_i v_i'' + w_i''}{v_i'' + w_i''} \tag{10}$$

If $s_i = 1$, then there is full offset. The same holds if utility is quasilinear - $v_i'' = 0$. We put off analysis of case C until after we consider income redistribution.

E…Comparative statics of income redistribution

We assume no government provision and lump-sum income redistribution:

$$t_1 + t_2 = 0 \tag{11}$$

Equilibrium is now:

$$v_1'\left[b_1 - t_1 - g_1\right] \geq w_1'\left[g_1 + g_2\right] \tag{12}$$
$$v_2'\left[b_2 + t_1 - g_2\right] \geq w_2'\left[g_1 + g_2\right] \tag{13}$$

Assume taxes are small enough not to change the type of equilibrium. Then, in case C, there is *no effect on the real allocation*.

This implies that there is full offset to government provision financed by contributors, assuming we stay with equilibrium of type C.

There would be an effect once we cross the border into type B. In type B, the critical question is whether redistribution is from or to contributors.

F…Multiple public goods

With multiple public goods and many people, we would expect most people to contribute to only one good. If there are multiple contributions for agent 1, for example, we have the first order condition (with w_{ik} being the utility of agent i with respect to public good k).

$$v_1'\left[b_1 - g_{11} - g_{12}\right] = w_{11}'\left[g_{11} + g_{21}\right] = w_{12}'\left[g_{12} + g_{22}\right] \tag{14}$$

If the budget of the agent is small relative to public good spending on any good, then we could start with marginal utilities assuming no contribution: $w_{ik}'\left[G + \sum_{j \neq i} g_{jk}\right]$. With possible contributions small, the marginal utilities do not change noticeably, and all of the contribution is made to the charity with the highest marginal utility at zero contribution.