Introduction to Political Economy 14.770
Problem Set 1 Solutions

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Question 1

Recall Arrow's impossibility theorem which states that if a social ordering is transitive, weakly Paretian and satisfies independence from irrelevant alternatives, then it is dictatorial.

1. *Consider a society with two individuals 1 and 2 and three choices, a, b, and c. For the purposes of this exercise, only consider strict individual and social orderings (i.e., no indifference allowed). Suppose that the preferences of the first agent are given by abc (short for $a \succ b \succ c$, i.e., a strictly preferred to b, strictly preferred to c). Consider the six possible preference orderings of the second individual, i.e., $s_2 \in \{abc, acb, bac, ...\}$, etc.. Define a social ordering as a mapping from the preferences of the second agent (given the preferences of the first) into a social ranking of the three outcomes, i.e., some function f such that the social ranking is $s = f(s_2)$. Illustrate the Arrow impossibility theorem using this example.*

   Suppose we have a transitive and Weakly Paretian social ordering ϕ which also satisfies the Independence of Irrelevant Alternatives (IIA). Let $\mathfrak{R}$ denote the set of strict orders on $\mathcal{P} = \{a, b, c\}$, i.e.

   $$\mathfrak{R} = \{abc, acb, bac, bca, cab, cba\}$$

   Here, $\phi : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$, and define $f : \mathfrak{R} \to \mathfrak{R}$ as

   $$f(\rho) = \phi(abc, s_2)$$

   for all $s_2 \in \mathfrak{R}$.

   Our claim is that we must have one of the two following cases:

(a) $f(s_2) = abc$ for all $s_2 \in \mathfrak{R}$ (in this case 1 is the dictator), or,

(b) $f(s_2) = s_2$ for all $s_2 \in \mathfrak{R}$ (in this case 2 is the dictator).

I will start this illustration by observing that, since ϕ is Weakly Paretian, we must have:

- $\phi(abc, abc) = abc \Rightarrow f(abc) = abc$
- $\phi(abc, acb) \in \{abc, acb\} \Rightarrow f(abc) \in \{abc, acb\}$
- $\phi(abc, bac) \in \{abc, bac\} \Rightarrow f(bac) \in \{abc, bac\}$
- $\phi(abc, bca) \in \{abc, bac, bca\} \Rightarrow f(bca) \in \{abc, bac, bca\}$
- $\phi(abc, cab) \in \{abc, acb, cab\} \Rightarrow f(cab) \in \{abc, acb, cab\}$

Perhaps an easier way to demonstrate this idea is the following table (where xy denotes that x must be ranked above y by the social ordering):

s_2	abc	acb	bac	bca	cab	cba
$f(s_2)$	abc	ab, ac	ac, bc	bc	ab	

I will investigate two possibilities:

(a) Assume $f(acb) = abc$.

- Note that individual 1 prefers b over c, individual prefers c over b and the social ordering ranks b over c. By IIA, the social ordering must rank b over c whenever this happens. Consequently, we must have $f(cab) = abc$ and $f(cba) \in \{abc, bac, bca\}$. We end up with the following table:

s_2	abc	acb	bac	bca	cab	cba
$f(s_2)$	abc	abc	ac, bc	bc	abc	bc

- Realize that $f(cab) = abc$, so when individual 1 prefers a over c and individual prefers c over a, the social ordering ranks a over c. By IIA, the social ordering must rank a over c whenever this happens. Consequently, we must have $f(bca) \in \{abc, bac\}$ and $f(cba) \in \{abc, bac\}$. We end up with the following table:

s_2	abc	acb	bac	bca	cab	cba
$f(s_2)$	abc	abc	ac, bc	ac, bc	abc	ac, bc

- Now assume, towards a contradiction, that $f(bac) = bac$. By IIA applied to a and b, whenever individual 1 prefers a over b and individual 2 prefers b over a, the social ordering must rank b over a. But then, consider the social ordering $\phi(acb, cba)$. Since individual 1 prefers a over b and individual 2 prefers b over a, the social ordering must rank b over a. Since individual 1 prefers a over c and individual 2 prefers c over a, the social ordering must rank a over c. Combining these, we must have:

$$\phi(acb, cba) = bac$$

which contradicts with ϕ being Weakly Paretian (both individuals prefer c over b but the social ordering ranks b over c).
- We conclude that we must have $f(bac) = abc$, so, by IIA applied to a and b, we must have: $f(bca) = f(cba) = abc$ as well.

s_2	abc	acb	bac	bca	cab	cba
$f(s_2)$	abc	abc	abc	abc	abc	abc

We conclude that when $f(acb) = abc$, we must have: $f(s_2) = abc$ for all $s_2 \in \mathfrak{R}$.

(b) Assume $f(acb) = acb$. You should have a good grasp of the idea by now, so I'll not repeat it step-by-step here. Very briefly,

- IIA applied to b and c implies $f(bac) \in \{acb, cab\}$, $f(cab) \in \{acb, cab\}$ and $f(cba) \in \{acb, cab, cba\}$.
- Assume, towards a contradiction, that $f(bac) = abc$. By IIA applied to a and b, we must have $f(bca) = abc$. Then, by IIA applied to a and c and IIA applied to b and c, we must have $\phi(bac, cba) = acb$, a contradiction to ϕ being Weakly Paretian. We conclude that we must have $f(bac) = bac$.
- By IIA applied to a and b, we must have: $f(bca) \in \{bac, bca\}$ and $f(cba) = cba$.
- By IIA applied to a and c, we must have: $f(s_2) = s_2$ for all $s_2 \in \mathfrak{R}$.

If you feel like you need a clearer guide, Section 2.1 of Austen-Smith and Banks' "Positive Political Theory I" has an in-depth discussion of this two-individuals, three-alternatives example.

2. *Now suppose we have the following aggregation rule: individual 1 will (sincerely) rank the three outcomes, his first choice will get 6 votes, the second 3 votes, the third 1 vote. Individual 2 will do the same, his first choice will get 8 votes, the second 4 votes, and the third 0 vote. The three choices are ranked according to the total number of votes. Which of the axioms of the Arrow's impossibility theorem does this aggregation rule violate?*

   This aggregation rule, commonly known as the Borda rule, violates the Independence of Irrelevant Alternatives (IIA).

   Consider the following example: individual 1 has preferences *abc* and individual 2 has preferences *bac*. It is trivial to check that the aggregation rule yields *bac* in this case. Now, consider the case where individual 1 has preferences *acb* and individual 2 has preferences *bac*. The aggregation rule yields *abc* in this case. Note that the relative ordering of a vs b has not changed in either agent's preferences in both cases, but the ordering given by the aggregation rule is changed, violating IIA.

3. *With the above voting rule, show that for a certain configuration of preferences, either agent has an incentive to distort his true ranking (i.e., not vote sincerely).*

   Assume that the social alternative which is ranked first by the aggregation rule is chosen. Consider the example in the previous part, where individual 1 has preferences *abc*, individual 2 has preferences *bac*, and b is chosen. Here, individual 1 can deviate and submit *acb*, ensuring that a is chosen instead. But when individual 1 submits *acb*, individual 2 can deviate and submit *bca*, ensuring that b is chosen instead of a.

   This non-strategy-proofness is not surprising: it is a natural implication of the Gibbard-Satterthwaite Theorem. (You can check Recitation 1 Notes if you need a refresher.)

4. *Now consider a society consisting of three individuals, with preferences given by:*

$$\begin{array}{ll} 1 & a \succ b \succ c \\ 2 & c \succ a \succ b \\ 3 & b \succ c \succ a \end{array}$$

   Consider a series of pairwise votes between the alternatives. Show that when agents vote sincerely, the resulting social ordering will be *"intransitive". Relate this to the Arrow's impossibility theorem.*

Consider the following set of pairwise votes:

- When a goes against b, a wins.
- When a goes against c, c wins.
- When b goes against c, b wins.

Therefore, the social ordering induced by pairwise voting rule yields $a \succ b \succ c \succ a$, violating transitivity. This is not surprising given Arrow's Impossibility Theorem, indeed: it is trivial to check that the pairwise voting rule is weakly Paretian, satisfies IIA and non-dictatorial. By Arrow's Impossibility Theorem, it must be intransitive (i.e. there is a preference profile for which the pairwise voting rule yields an intransitive social ordering).

5. *Show that if the preferences of the second agent are changed to $b \succ a \succ c$, the social ordering is no longer "intransitive". Relate this to "single-peaked preferences".*

   In this case,

   - When a goes against b, b wins.
   - When a goes against c, a wins.
   - When b goes against c, b wins.

   So the social ordering induced by pairwise voting rule yields $b \succ a \succ c$, a transitive one.

   Note that these preferences are single-peaked when according to the (intuitive) ordering with a being the most left-wing policy and c being the most right-wing one. This restriction on preference domain allows us to invoke the median voter theorem, which guarantees that a Condorcet winner exists (b in this case).

6. *Explain intuitively why single-peaked preferences are sufficient to ensure that there will not be intransitive social orderings. How does this relate to the Arrow's impossibility theorem?*

   There are of course more than one correct answers to this question. Mine would be: the restriction to single-peaked preferences also makes the "support" a policy receives a single-peaked function. That is, a policy predictably becomes more popular if we are moving towards the median, and it loses support if e move away from the median.

Question 2

1. *Consider the example of a three-person three-policy society with preferences*

$$\begin{array}{ll} 1 & a \succ b \succ c \\ 2 & b \succ c \succ a \\ 3 & c \succ b \succ a \end{array}$$

Voting is dynamic: first, there is a vote between a and b. Then, the winner goes against c, and the winner of this contest is the social choice. Find the subgame perfect Nash equilibrium with weakly undominated strategies within each stage in this two-stage game.

First, some general commentary about the "point" of this question. Begin by realizing that the preferences are single-peaked with respect to the alphabetical order (a is the left-most and c is the right-most policy). Following the discussion in class, we know that there is a Condorcet winner in this case: namely, it is the most preferred choice of the voter with the median bliss point. The median voter in this question turns out to be player 3, and her most preferred policy is b – it is the "moderate" policy among the three. It is trivial to check that b is the Condorcet winner, and we would expect any reasonable voting rule to implement b in this case. Nevertheless, as also discussed in the lecture (see page 27 of Lecture 1 and 2 notes) things tend to get complicated when voters are voting strategically. This question attempts to argue that sequential voting implements the reasonable policy (the Condorcet winner) even when voters are strategic. Belief in electoral systems restored! :)[1]

I would argue that this turned out to be a less-spectacular-than-intended question because the definition of *weakly undominated strategies within each stage* is not clear.[2] Now, following our discussion in class, in a one-stage voting game the only weakly undominated strategy is voting sincerely (because a strategic player should condition on herself being pivotal, and whenever she is pivotal she should vote for her best option, i.e. vote sincerely). In a dynamic (multi-stage) game, though, this definition requires a little bit more elaboration. In retrospect, I believe we should have asked for a *Subgame Perfect Nash Equilibrium with strategies which survive the Iterated Elimination of*

[1] This belief only lasts until Lectures 6 and 7, though.

[2] I've also had a hard time figuring out the exact distinction and the meaning behind this wording.

Weakly Dominated Strategies – semantically a much better choice, especially for a theorist. I'll provide the solution assuming this wording. Below, I'm trying to be as pedagogical as possible and I'm providing the answers for (i) Subgame Perfect Nash Equilibrium, (ii) Subgame Perfect Nash Equilibrium with weakly undominated strategies, and (iii) Subgame Perfect Nash Equilibrium with strategies which survive the Iterated Elimination of Weakly Dominated Strategies.

Some basics first: Following the hint given in the question, a (pure) **strategy** for a player $i \in \{1, 2, 3\}$ is a 3-tuple:

$$s_i \in \{a, b\} \times \{a, c\} \times \{b, c\}$$

where the first element denotes the vote in the first stage, the second element denotes the vote in the subgame where a wins in the first stage, and the third element denotes the vote in the subgame where b wins in the first stage.

Let S denote the set of all possible strategies for a player (there are eight of them!). A **strategy profile** is $(s_1, s_2, s_3) \in S \times S \times S$. Given a player $i \in \{1, 2, 3\}$, let $s_{-i} \in S \times S$ denote the strategy profile of the remaining profiles.

Let $\phi(s_1, s_2, s_3)$ denote the outcome of the voting procedure defined in the question. Formally, it is a well-defined function:

$$\phi : S \times S \times S \to \{a, b, c\}$$

(i) Begin with the case where we're only looking for a Subgame Perfect Nash Equilibrium. I know this wasn't asked in the problem set, but it's useful to see what goes wrong when people are strategic and we're imposing a not-so-stringent equilibrium concept.

The basic issue you need to realize in this case is: many "crazy" equilibria can happen when we're only looking for subgame perfection. For instance, the following strategy profile (s_1^*, s_2^*, s^*3) is a Subgame Perfect Nash Equilibrium, but it doesn't yield the Condorcet winner:

$$s_1^* = (a, a, c)$$
$$s_2^* = (a, a, c)$$
$$s_3^* = (a, a, c)$$

This is a Subgame Perfect Nash Equilibrium simply because none of the voters are pivotal at any stage, so unilaterally changing their votes would not change the outcome. Consequently, none of the players have a strictly profitable deviation. Note that the implemented policy is a, the left-most policy. This is a weird equilibrium because we cooked it up such that none of the players really "care" about the equilibrium: they realize they cannot change the outcome. For instance, player 3 really hates this outcome but still votes for a in the second round, just because she knows a will win regardless of her vote. If there's even a tiny bit of possibility that player 3 is the pivotal voter in the second round, she would vote for b instead. In other words, even though voting for b is player 3's weakly dominant strategy in the last round, she doesn't use this strategy because she knows the gains from this strategy will never materialize.

One way to overcome this weird implication is to assume that players never use a weakly dominated strategy, which is what we do next.

(ii) Now, let's assume that each player uses a weakly undominated strategy. Formally, a strategy $s_i \in S$ for player $i \in \{1,2,3\}$ is **weakly dominated** if there exists another strategy $s_i' \in S$ such that

$$\phi(s_i', s_{-i}) \succeq_i \phi(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S \times S$$

with

$$\phi(s_i', s_{-i}) \succ_i \phi(s_i, s_{-i}) \quad \text{for some } s_{-i} \in S \times S$$

A weakly undominated strategy is simply a strategy which is not weakly dominated.

It should be easy to see (following our discussion in class) that any player which uses a weakly undominated strategy should be voting sincerely in the last stage. To illustrate, take player 1 and assume she uses a strategy $s_1 = (s_{1,1}, c, s_{1,3})$ – that is, she doesn't vote sincerely in the subgame where a wins in the first round. Take the alternative strategy $s_1' = (s_{1,1}, a, s_{1,3})$, i.e. the strategy of doing the same thing except voting for a in the second round if a wins. You should be able to see that s_1 is weakly dominated by s_1'.[3] By the same vein, any strategy of the form $s_1 = \{s_{1,1}, s_{1,2}, a\}$

[3] They yield the same payoff if the other two players vote for the same alternative in the subgame where a wins in the first round, or, if b wins the first round. Nevertheless, s_1'

is weakly dominated by $s'_1 = \{s_{1,1}, s_{1,2}, c\}$. One can check that no other strategy is weakly dominated.

It follows that the set of weakly undominated strategies S_i^u for player $i \in \{1, 2, 3\}$ are defined as:

$$S_1^u = \{a, b\} \times \{a\} \times \{b\}$$
$$S_2^u = \{a, b\} \times \{c\} \times \{b\}$$
$$S_3^u = \{a, b\} \times \{c\} \times \{c\}$$

We have reduced the set of strategies for each player to two per player, rather than eight per player, which is a big deal! Now, we can calculate the Subgame Perfect Equilibrium of the game assuming that player $i \in \{1, 2, 3\}$ uses a strategy in S_i^u. It turns out that this reduction is not sufficient to get the desired result, nevertheless. In particular, you can check that the strategy profile $(s_1^*, s_2^*, s_3^*) \in S_1^u \times S_2^u \times S_3^u$ is a Subgame Perfect Nash Equilibrium in weakly undominated strategies, but it still doesn't yield the Condorcet winner:

$$s_1^* = (a, a, b)$$
$$s_2^* = (a, c, b)$$
$$s_3^* = (a, c, c)$$

This is a Subgame Perfect Nash Equilibrium because, similar to the example we gave in previous part, none of the voters are pivotal in the first stage, so unilaterally changing their votes would not change the outcome. Consequently, none of the players have a strictly profitable deviation. Note that the implemented policy is c, the right-most policy. Note that the implausible implication from previous part still continues: player 1 hates outcome c, but still votes for outcome a (even though she correctly infers that it will lose to c in the second round) because she cannot change the outcome. One way to overcome this is assuming that players run one more round of elimination of weakly dominated strategies, which is what we do next.

yields a strictly better outcome for player 1 if a wins in the first round and the two other players vote for different alternatives in the subgame where a wins.

(iii) Let's now assume that each player uses a strategy which survives the Iterated Elimination of Weakly Dominated Strategies. Formally, this procedure is defined as:

Step 0: Define $S_i^0 = S$

Step 1: Define

$$\begin{aligned} S_i^1 = \{ s_i \in S_i^0 | \nexists s_i' \in S_i^0 \ & with \\ \phi(s_i', s_{-i}) \succeq_i \phi(s_i, s_{-i}) & \text{ for all } s_{-i} \in S_{-i}^0 \text{ and} \\ \phi(s_i', s_{-i}) \succ_i \phi(s_i, s_{-i}) & \text{ for some } s_{-i} \in S_{-i}^0 \} \end{aligned}$$

S_i^1 is the set of weakly undominated strategies: simply, $S_i^1 = S_i^u$.

Step $k+1$: Define

$$\begin{aligned} S_i^{k+1} = \{ s_i \in S_i^k | \nexists s_i' \in S_i^k \ & with \\ \phi(s_i', s_{-i}) \succeq_i \phi(s_i, s_{-i}) & \text{ for all } s_{-i} \in S_{-i}^k \text{ and} \\ \phi(s_i', s_{-i}) \succ_i \phi(s_i, s_{-i}) & \text{ for some } s_{-i} \in S_{-i}^k \} \end{aligned}$$

S_i^{k+1} is the set of strategies which are still weakly undominated when you know other players use something in S_{-i}^k.

Clearly, this process defines an iterated deletion weakly dominated strategies. Final step:

$$S_i^\infty = \cap_{k=0}^\infty S_i^k$$

Let's solve this game assuming that each player $i \in \{1, 2, 3\}$ uses a strategy in S_i^∞. The first round of elimination discussed in the previous part still applies, and we have:

$$\begin{aligned} S_1^1 &= \{a, b\} \times \{a\} \times \{b\} \\ S_2^1 &= \{a, b\} \times \{c\} \times \{b\} \\ S_3^1 &= \{a, b\} \times \{c\} \times \{c\} \end{aligned}$$

Thefefore, each player realizes that:

- If a wins in the first stage, c will win in the second stage.
- If b wins in the first stage, b will win in the second stage.

This is a strong observation – it implies that each agent realizes that the voting they're having in the first round is not between a and b. It is, rather, a choice between c and b. Consequently, even if a is player 1's favorite choice, she has incentives to refrain from voting for a, and vote for b instead (because she prefers b to c).

Let's now continue with the second round of iteration. You can see that for each player $i \in \{1, 2, 3\}$, one of the strategies in S_i^1 is weakly dominated. To illustrate, take player 1 and compare the strategies $s_1 = (a, a, b)$ and $s_1' = (b, a, b)$, assuming that $s_2 \in S_2^1$ and $s_3 \in S_3^1$. In this case, s_1 is weakly dominated by s_1'.[4] It follows that the set of strategies which survive the Iterated Elimination of Weakly Dominated Strategies:

$$S_1^\infty = \{(b, a, b)\}$$
$$S_2^\infty = \{(b, c, b)\}$$
$$S_3^\infty = \{(a, c, c)\}$$

The following is a Subgame Perfect Equilibrium in strategies which survive the Iterated Elimination of Weakly Dominated Strategies:

$$s_1^* = (b, a, b)$$
$$s_2^* = (b, c, b)$$
$$s_3^* = (a, c, c)$$

Realize that agents don't vote sincerely in the first stage: both 1 and 3 lie about their preferences. Nevertheless, the implemented policy is still b, the Condorcet winner.

2. *Suppose a generalization whereby there are finite number of policies, $Q = \{q_1, q_2, ..., q_N\}$ and M agents (which you can take to be an odd number for simplicity). Voting takes $N - 1$ stages. In the first stage, there is a vote between q_1 and q_2. In the second stage, there is a vote between the winner of the first stage and q_3, until we have a final vote against q_N. The winner of the final vote is the policy choice of the society. Prove that if preferences of all agents are single peaked (with a unique bliss point for each), then the unique subgame perfect*

[4] They yield the same payoff if the other two players vote for the same alternative in the first round. Nevertheless, s_1' yields a strictly better outcome for player 1 if the two other players vote for different alternatives in the first round.

Nash equilibrium with weakly undominated strategies within each stage implements the bliss point of the median voter.

Once again, rather than *weakly undominated strategies within each stage*, I'll refer to *strategies which survive the Iterated Elimination of Weakly Dominated Strategies*. Assuming you get the gist of the idea in part 1, I'll not provide the whole proof here, but rather give the general idea.

The proof is a generalization of the idea presented in part 1. Note that in this case a strategy consists of: (i) a vote in $\{q_1, q_2\}$ in Round 1, (ii) a vote in $\{q_1, q_3\}$ if q_1 advances in Round 1, (iii) a vote in $\{q_2, q_3\}$ if q_2 advances in Round 1, (iv) a vote in $\{q_1, q_4\}$ if q_1 advances in Round 2... So it's a pretty complicated object. For conciseness, I'll suppress the notation and talk about Round k strategies in general. Furthermore, for any $l < k$, given that q_l advances to Round k, it doesn't really matter which alternative q_l beats in Round $l-1$ (i.e. it doesn't matter which alternative advanced to compete with q_l as long as it lost to q_l), so this saves us from some unnecessary complications in defining policy space. Shortly, we can refer to $s_{l,k} \in \{q_l, q_k\}$ as the vote if q_l advances to Round $k-1$.

Since preferences are single-peaked, we know that there is a Condorcet winner. Let q_j, with $j \in \{1, \dots, N\}$, denote the Condorcet winner.

Begin with the eliminations in the last round, Round $k-1$. Consider the strategies $s_{j,N}$, i.e. the strategies that agents will use if j advances to Round $N-1$. Since this is the last round, by the same argument as given in part 1, the agents who use weakly undominated strategies vote sincerely. But since q_j is the Condorcet winner, by definition, q_j receives more votes than q_N and wins. The similar calculations are done for every possible alternative advancing to Round $N-1$, and the winners are determined for every contingency. Let $w(i, N) \in \{q_i, q_N\}$ denote the winner determined by this process, as a result of weakly undominated strategies $s_{i,N}$. Since we assume that M is odd, $w(i, N)$ is well-defined for each $i \in \{1, \dots, N-1\}$.

Now, let's consider the next round of eliminations. In particular, consider the strategies $s_{j,N-1}$. In principle, there is no requirement to expect sincere voting at this stage – this is because, similar to the observation we made in part 1, strategic agents realize that it is not a voting between q_j and q_{N-1}; rather, it is a voting between q_j and $w(N-1, N)$. Nevertheless, regardless of which alternative $w(N-1, N)$

is, q_j wins in this round because it is the Condorcet winner. Thus q_j also wins in this round, even if people may not vote sincerely.

One can continue this induction to realize that in Rounds j through $N-1$, q_j wins against every alternative q_l, even though people realize that they're not indeed voting between q_j and q_l. (This is simply as an implication of q_j being the Condorcet winner, and note the crucial role of single-peaked preferences here!) At Round $j-1$, for the strategy $s_{i,j}$ for $i < j$, each agent also realizes that the vote is not between q_i and q_j, but rather between the eventual candidate which wins in the last Round if q_i advances, and q_j (which, as established, wins in the following rounds). But once again, being the Condorcet winner, q_j wins in this case as well. This proves that in Round $j-1$, regardless of the alternative which proceedss to this round, q_j wins and advances, beating every other alternative afterwards and thus being the eventual winner. We conclude that the procedure implements q_j, the bliss point of the median voter.

Question 3

Consider party competition in a society consisting of a continuum of mass 1 of agents, where the set of agents is $\mathcal{H}$. The policy space is the $[0,1]$ interval and assume that preferences are single-peaked. In particular, if an agent $i \in \mathcal{H}$ has bliss point b_i, her utility from policy $q \in [0,1]$ is:

$$u(b_i, q) = -|b_i - q|$$

Finally, assume that the bliss points are uniformly distributed over this space.

1. *To start with, suppose that there are two parties, A and B. They both would like to maximize the probability of coming to power. The game involves both parties simultaneously announcing $q_A \in [0,1]$ and $q_B \in [0,1]$, and then voters voting for one of the two parties. The platform of the party with most votes gets implemented. Determine the equilibrium of this game. How would the result be different if the parties maximized their vote share rather than the probability of coming to power?*

   The question is silent on what the agents who are indifferent between the two policies do (do they randomize or do they vote for one of the parties?), so I'll discuss both cases. As the starting point, let's assume that a voter who is indifferent between two policies randomizes between the two parties – and votes for party A with probability $\pi \in (0,1)$.[5] I'll discuss the remaining cases ($\pi \in \{0,1\}$) later.

Assume that the parties are maximizing the probability of coming to power. We begin by observing that we cannot have an equilibrium with $q_A \neq q_B$. To see why, suppose, towards a contradiction, that we have an equilibrium with $q_A \neq q_B$.

- If there is a tie, one of the parties can deviate to $q_i' = \frac{1}{2}$ and increase the probability of winning to one. (Surely the other party is not proposing $\frac{1}{2}$ – we would not have a tie in that case.)
- If one of the parties is losing, then the losing party can deviate to $\frac{1}{2}$ and increase the probability of winning to π, $1-\pi$ or 1, each strictly positive.

It turns out that in any equilibrium, we need to have $q_A = q_B = q$. Next, we claim that we cannot have $q \neq \frac{1}{2}$. To see why, suppose, towards a contradiction, that we have an equilibrium with $q \neq \frac{1}{2}$. In such a case, Party A is winning with probability $\pi \in (0,1)$. But then, either party can deviate to $\frac{1}{2}$ and win with probability one.

It is also easy to see that $q_A = q_B = \frac{1}{2}$ is an equilibrium – we conclude that this is the unique equilibrium. This is simply the Downsian Policy Convergence we discussed in Lectures 1 and 2: both parties cater to the median voter.

What happens with $\pi \in \{0,1\}$? In such a case, in equilibrium, the winning party is still located in $q = \frac{1}{2}$, but the losing party can offer any policy in $[0,1]$ (it will lose with probability one no matter what policy it offers, so it is indifferent). Even though the strong form of Downsian Policy Convergence does not hold, we still have the median voter's favorite policy being implemented in equilibrium – so we're not that far away. (It is also worth pointing that $\pi \in \{0,1\}$ is a pathological corner case which we shouldn't worry much about.)

Is there much difference when parties maximize their vote shares rather than the probability of coming to power? I'd argue not. Using the same arguments, one can show that when $\pi = \frac{1}{2}$, the Downsian Policy Convergence still holds and the unique equilibrium is $q_A = q_B = \frac{1}{2}$.

[5] Assuming that a fraction π of voters always vote for A when indifferent, and the rest voting for B when indifferent also works. Also note that this only matters when $q_A = q_B$. When $q_A \neq q_B$, the measure of voters who are indifferent between the policies is zero.

When $\pi \neq \frac{1}{2}$, there is an existence problem due to the lower hemi-continuity of the payoff function of the "less popular" party (it wants to get as close to $\frac{1}{2}$, but does not want to offer $\frac{1}{2}$). But the median voter's favorite policy is still implemented, and this is still likely to be considered as a pathological case we should worry less about.

2. *Now assume that there are three parties, simultaneously announcing their policies $q_A \in [0,1]$, $q_B \in [0,1]$, and $q_C \in [0,1]$, and the platform of the party with most votes is implemented. Assume that parties maximize the probability of coming to power. Characterize all pure strategy equilibria.*

   From now on, I will assume that whenever a voter is indifferent among two or more parties, she randomizes between them with equal probabilities – otherwise the analysis gets cumbersome due to reasons discussed above.

   The following are solutions I generously borrowed from Horacio Larreguy, one of the previous TAs of this class.

   Start as before with potential equilibria of the kind $q_A = q_B = q_C = q^*$. There are two subcases to consider: $q^* \neq 1/2$ and $q^* = 1/2$. In this case we have that the probability of winning of party i is $1/3$. Then if $q^* \neq 1/2$; i could deviate to $q_i' = 1/2$ and win the election with probability 1 (it gets at least one half of the votes, while the other parties get at most $1/4$ each). If $q^* = 1/2$, any party could deviate to $q' = q^* + \varepsilon$ and get almost $1/2$ of the vote share and win the election with probability 1.

   Now let's analyze potential equilibria where, without loss of generality, we have $q_A < q_B < q_C$. In this case the vote share of parties $S_A = \frac{q_A+q_B}{2}$; $S_B = \frac{q_C-q_A}{2}$, $S_C = \frac{q_C+q_B}{2}$. Now suppose there is a tie between the three parties, i.e. $S_A = S_B = S_C$. In this case party could move ε to the right and win with probability 1, hence there can not be an equilibrium with $q_A < q_B < q_C$; and $S_A = S_B = S_C$.

   Now assume that two parties are tying and one is losing. If B and A or C are tying, then the party on the extreme can move closer to B and win with probability 1. If A and C are tying, and B is losing, either A or C could move closer to B and win with probability 1.

   Now suppose one party is winning with probability 1. If party B is the one winning, then either party A or party C could move to $q_B \pm \varepsilon$ and win with probability 1 (by getting more than half the share). Now assume that party A or party C is winning with probability 1. Clearly, if $q_A > 1/2$; party B could move to $1/2$ and win with probability 1. If $q_C < 1/2$ a similar argument could be used, B could move to $1/2$ and win with probability 1.

Then we are left only with cases with $q_A < 1/2; q_C > 1/2; q_B \in (q_A; q_C)$. Take the cases where A is winning. Assume that $q_B < 1/2$. In this case party C could move to $1/2$ and win with probability 1. Now take the case where $q_B \geq 1/2$. For party A to be winning we need to have:

$$\begin{aligned} \frac{q_A + q_B}{2} &> \frac{q_C - q_A}{2} \\ \frac{q_A + q_B}{2} &> 1 - \frac{q_C + q_B}{2} \end{aligned}$$

which are equivalent to

$$\begin{aligned} \frac{q_B}{2} &> \frac{q_C}{2} - q_A \\ \frac{q_A + q_C}{2} &> 1 - q_B \end{aligned}$$

Now let's consider potential deviations. Party B has two potential deviations. The first is to move to q_A and get the same votes as party A. This is not profitable if

$$\begin{aligned} 1 - \frac{q_C + q_A}{2} &> \frac{q_C + q_A}{4} \\ &\Leftrightarrow \frac{q_C + q_A}{2} < \frac{2}{3} \end{aligned}$$

The other deviation is to move to $q_A - \varepsilon$. This is not profitable if

$$\begin{aligned} q_A &< 1 - \frac{q_C + q_A}{2} \\ \frac{q_C + q_A}{2} &< 1 - q_A \end{aligned}$$

we must have then that

$$1 - q_B < \frac{q_C + q_A}{2} < \min\left\{1 - q_A, \frac{2}{3}\right\}$$

Now take the deviations by C. The first deviation it could make is move to $q_B + \varepsilon$. This is not profitable if

$$\begin{aligned} 1 - q_B &< \frac{q_A + q_B}{2} \\ 2 - 3q_B &< q_A \end{aligned}$$

C could also jump into A. This is not profitable if

$$\frac{q_B + q_A}{2} < \frac{2}{3}$$

Finally we have

$$\frac{q_B + q_A}{2} < 1 - q_A$$

Then, we can find an equilibrium of this kind if the following holds:

$$\begin{aligned} 1 - q_B &< \frac{q_B + q_A}{2} < \frac{q_C + q_A}{2} < \min\left\{1 - q_A, \frac{2}{3}\right\} \\ q_A &> \frac{q_C}{2} - \frac{q_B}{2} \\ q_B &\geq 1/2 \end{aligned}$$

Take an example with $q_A = 1/3$, $q_C = 7/8$, $q_B = 3/4$.

Notice

$$\begin{aligned} \frac{1}{4} &< \frac{1}{2} < \frac{29}{48} < \frac{2}{3} \\ \frac{1}{3} &> \frac{7}{16} - \frac{6}{16} = 1/16 \end{aligned}$$

Hence, we have equilibria of this kind and the symmetrical case with C winning.

We have the last case to consider $q_A < q_B = q_C$ (and the symmetrical case with $q_A = q_B < q_C$). Here there are three possibilities, either A wins alone, a three party tie, or a two party tie (B and C). First notice that A can't be losing, otherwise it can move to q_C and create a tie. Now assume that A is winning with probability one. It has to be that there is a configuration of the kind $q_A < 1/2 < q_B = q_C$. This is only possible if $(q_A + q_B)/2 > 1/3$. Also for B,C to be unwilling to move it must be the following restrictions hold

$$\frac{1}{3} < \frac{q_A + q_B}{2} < \min\left\{1 - q_A, \frac{2}{3}\right\}$$

Take again the example of $q_A = 1/3,\ q_B = 3/5$ in which this holds.

Finally we have a three party tie with $q_A < q_B = q_C$ party A can move to the right and win with probability 1.

Hence there are two types of equilibria of this game

(a) Two parties with $q_i = q_j < 1/2\,(> 1/2)$ and $q_k > 1/2\,(< 1/2)$ with party k winning with probability 1.

(b) Parties locating in different positions with $q_i > q_j > q_k$, with one extreme party winning with probability 1. If party k wins then $q_i < 1/2, q_j < 1/2, q_k > 1/2$. If party i is winning, then $q_i < 1/2, q_j > 1/2, q_k > 1/2$.

3. *Now assume that the three parties maximize their vote shares. Prove that there exists no pure strategy equilibrium. Characterize the mixed strategy equilibrium (Hint: assume the same symmetric probability distribution for two parties, and make sure that given these distributions, the third party is indifferent over all policies in the support of the distribution).*

   Once again, courtesy of Horacio Larreguy.

   It is straightforward to check that once parties maximize vote shares the above equilibria breakdown and there is no pure strategy Nash equilibrium of the game.

   Now, to construct the mixed strategy equilibrium, suppose there is a symmetric equilibrium with $f(q)$ being the mixing function in equilibrium, with support $\left[\underline{q}, \overline{q}\right]$. Furthermore assume the support of f is $[\alpha, 1-\alpha]$ with $\alpha \in [0, 1/2]$. The cumulative distribution is the $F(x) = \int_\alpha^x f(z)\,dz$.

   Now, without loss of generality, assume that party C chooses location z, the payoff she gets are:

$$\begin{aligned} S(z) &= 2\int_\alpha^z f(x)\,F(x)\left(1-\frac{z+x}{2}\right)dx \\ &\quad + \int_\alpha^z \int_z^{1-\alpha} f(x)\,f(y)\,\frac{y-x}{2}\,dy\,dx \\ &\quad + 2\int_z^{1-\alpha} f(x)\,(1-F(x))\left(\frac{z+x}{2}\right)dx \end{aligned}$$

   where the first integral is the expected payoff when party A's x and party B's y are located to the left of party C's z, WLOG the second integral corresponds to the case where party C's z is in between of party A's x and party B's y, and the last integral is the expected payoff when party A's x and party B's y are located to the right of party C's z.

Then, taking the differential of the expression above we get

$$\begin{aligned} \frac{\partial S(z)}{\partial z} &= 2f(z)F(z)(1-z) - \int_{\alpha}^{z} f(x)F(x)\,dx \\ &+ f(z)\int_{z}^{1-\alpha} f(y)(y-x)\,dy - f(z)\int_{\alpha}^{z} f(x)(z-x)\,dx \\ &-2zf(z)(1-F(z)) + \int_{z}^{1-\alpha} f(x)(1-F(x))\,dx \end{aligned}$$

Using the fact that

$$\begin{aligned} \int f(x)F(x)\,dx &= \frac{1}{2}F^2(x) \\ \int_{\alpha}^{1-\alpha} xf(x)\,dx &= \frac{1}{2} \end{aligned}$$

we can rewrite the expression above as

$$\frac{\partial S(z)}{\partial z} = \left(\frac{1}{2} - F(z)\right) + f(z)\left(z\left(\frac{1}{2} - z\right) - 2\left(\frac{1}{2} - F(z)\right)\right)$$

For this to be a mixed strategy equilibrium we must have $\frac{\partial S(z)}{\partial z} = 0$, for every $z \in [\alpha, 1-\alpha]$. That is, he is indifferent.

Define

$$h(z) = \frac{\frac{1}{2} - F(z)}{\frac{1}{2} - z}, h'(z) = \frac{h(z) - f(z)}{\frac{1}{2} - z}$$

Then we have

$$\frac{\partial S(z)}{\partial z} = 0 \Leftrightarrow h'(2h - z)\left(\frac{1}{2} - z\right) = 2h(h-2)$$

which yields

$$h^3(h-2) = K\left(\frac{1}{2} - z\right)^{-4}$$

where K is a constant. Take $K = 0$, we have

$$h = 2 = \frac{\frac{1}{2} - F(z)}{\frac{1}{2} - z}$$

which implies

$$F(z) = 2\left(z - \frac{1}{4}\right)$$

For this to be a CDF we need $F(\alpha) = 0 \Leftrightarrow \alpha = 1/4$. We can check, $F(1-\alpha) = 1$.

Finally we need to show that $S(z) < S^* \ \forall z \notin [1/4, 3/4]$ where S^* is the share they get in equilibrium (equal to the share of playing any of the z in the support)

$$\begin{aligned} S(z) &= 2\int_{1/4}^{3/4} f(x)(1 - F(x))\left(\frac{z+x}{2}\right) dx \\ &= const + z\int_{1/4}^{3/4} f(x)(1 - F(x))\, dx \end{aligned}$$

since $f(x) > 0 \ \forall x$, and $(1 - F(x)) > 0 \ \forall x$, this is increasing in z which implies $S(1/4) > S(z) \ \forall z < 1/4$.

Similarly, take $z > 3/4$, then the share of party C is:

$$\begin{aligned} S(z) &= 2\int_{1/4}^{3/4} f(x)(1 - F(x))\left(1 - \frac{z+x}{2}\right) dx \\ &= const - z\int_{1/4}^{3/4} f(x)(1 - F(x))\, dx \end{aligned}$$

hence the share for values greater than the upper end of the support is lower than the equilibrium share.

Then, playing a mixed strategy $f \sim U[1/4, 3/4]$ is a symmetric mixed strategy equilibrium of the game.

Question 4:

Consider the following one-period economy populated by a mass 1 of agents. A fraction λ of these agents are capitalists, each owning capital k.

The remainder have only human capital, with human capital distribution $F(h)$. Output is produced in competitive markets, with aggregate production function

$$Y = K^{1-\alpha} H^{\alpha},$$

where uppercase letters denote total supplies. Assume that factor markets are competitive and denote the market clearing rental price of capital by r and that of human capital by w.

1. *Suppose that agents vote over a linear income tax, τ. Because of tax distortions, total tax revenue is*

$$Tax = (\tau - v(\tau)) \left(\lambda r k + (1-\lambda) w \int h dF(h) \right)$$

where $v(\tau)$ is strictly increasing and convex, with $v(0) = v'(0) = 0$ and $v'(1) = \infty$ (why are these conditions useful?). Tax revenues are redistributed lump sum. Find the ideal tax rate for each agent. Find conditions under which preferences are single peaked, and determine the equilibrium tax rate. How does the equilibrium tax rate change when k increases? How does it change when λ increases? Explain.

For a capitalist, the utility over tax rate $\tau \in [0,1]$ is:

$$u_k(\tau) := (1-\tau) r k + (\tau - v(\tau)) Y$$

Note that since $v(.)$ is convex, $u_k(\tau)$ is concave – which immediately establishes that capitalists have single-peaked preferences over τ. The bliss point (i.e. the most preferred tax rate) for a capitalist is the value of $\tau \in [0,1]$ which maximizes $u_k(\tau)$:

$$\tau_k^* := \arg \max_{\tau \in [0,1]} u_k(\tau)$$

To find it, take the first order condition:

$$\frac{\partial u_k(\tau)}{\partial \tau} = Y - rk - v'(\tau) Y$$

Realize that we have two possible cases:

- If $Y - rk \leq 0$, then $u_k(\tau)$ is decreasing everywhere and $\tau_k^* = 0$.

- If $Y - rk > 0$, then $u_k(\tau)$ is increasing in $\tau = 0$, so we have $\tau_k^* > 0$. Moreover, since $\lim_{\tau \to 1} v'(\tau) = \infty$, we know we must have $\tau_k^* < 1$. We conclude that we must have an interior solution given by:
$$\tau_k^* = v'^{-1}\left(1 - \frac{rk}{Y}\right)$$

For notational simplicity, let $g(x) := v'^{(}-1)(x)$. One crucial thing to note here is that, since $v(.)$ is a convex function, $v'(.)$ is increasing and so is $g(.)$.

Now, we have a bunch of simplifications. Using (i) that $Y = K^{1-\alpha}H^{\alpha}$, (ii) that the output is produced in competitive markets (so that $r = MPK = (1-\alpha)K^{-\alpha}H^{\alpha}$), and (iii) that $K = \lambda k$, one can show that:

- $Y - rk \leq 0 \Leftrightarrow \lambda \leq 1 - \alpha$
- $1 - \frac{rk}{Y} = 1 - \frac{1-\alpha}{\lambda}$

To sum, all capitalists have the same optimal tax rate given by:

$$\tau_k^* = \begin{cases} 0, & \text{if } \lambda \leq 1 - \alpha \\ g\left(1 - \frac{1-\alpha}{\lambda}\right), & \text{if } \lambda > 1 - \alpha \end{cases}$$

A couple of observations about the most preferred tax rate of a capitalist:

(a) It is equal to zero if $\lambda \leq 1 - \alpha$. Heuristically, λ is the population share of capitalists and $1 - \alpha$ is the share of national income distributed among the capitalists (a virtue of Cobb-Douglas production function). If there are fewer capitalists compared to the total income of capitalists, then capitalists are richer and don't want any redistribution.

(b) It is (weakly) increasing in λ. Again heuristically, if there are more capitalists around, there is less income per capitalists, so they demand more redistribution.

(c) It doesn't depend on k – I'd say that this is also a virtue of Cobb-Douglas production function: it adjusts factor prices nicely, so that capitalists are neutral towards having more or less capital.

Now, the same exercise for workers (holders of human capital). For a worker with human capital h, the utility over tax rate $\tau \in [0,1]$ is:

$$u_h(\tau) := (1-\tau)wh + (\tau - v(\tau))Y$$

Once again, convexity of $v(.)$ guarantees having single-peaked preferences over τ. The bliss point is:

$$\tau_h^* := \arg\max_{\tau \in [0,1]} u_h(\tau)$$

And the first order condition is:

$$\frac{\partial u_h(\tau)}{\partial \tau} = Y - wh - v'(\tau)Y$$

Once again, we have two possible cases, and making the substitutions along the same line gives us a simplified version:

$$\tau_h^* = \begin{cases} 0, & \text{if } h \geq \frac{H}{\alpha} \\ g\left(1 - \frac{\alpha h}{H}\right), & \text{if } h < \frac{H}{\alpha} \end{cases}$$

where, for the sake of closure, note that $H = (1-\lambda)\int h dF(h)$.

Some observations again:

(a) Unlike capitalists', a worker's most preferred tax rate depends on h. This is simply because workers have heterogeneous human capital levels – consequently, a worker should now take into account how she fares compared to the average human capital in the society, H.

(b) Once again, the bliss point is zero if $h \geq \frac{H}{\alpha}$. Heuristically, a worker with a higher h is richer, and she doesn't want redistribution if she is sufficiently rich compared to other workers.[6]

(c) (Keeping H constant) the bliss point is decreasing in h. The heuristics is the same: a richer worker demands less redistribution.

[6] It may be also useful to note that if each worker have the same human capital level h, this would reduce to: $1 - \gamma \leq \alpha$, much similar to the case with capitalists.

(d) It is (weakly) decreasing in λ. The intuition for this is similar to the same comparative statics with respect to capitalists, but it's less meaningful: most because when λ changes the "median" worker changes as well.

(e) It doesn't depend on k, for the same reasons.

Now, let's consider the equilibrium tax rate. We have already argued that the preferences are single-peaked, and by Median Voter Theorem, we know there is a Condorcet winner, given by the bliss point of the median individual (when they're ranked according to their bliss points). There are several cases:

- If $\lambda > \frac{1}{2}$, capitalists are the majority and the median voter will be a a capitalist. In this case,
  - If $\lambda \leq 1-\alpha$, $\tau^* = 0$. This is the case where capitalists are a crowded group, but also their share of national income is sufficiently large so they don't want any redistribution.
  - If $\lambda > 1-\alpha$, $\tau^* = g\left(1 - \frac{1-\alpha}{\lambda}\right) \in (0,1)$.
- If $\lambda < \frac{1}{2}$, workers are the majority. In this case,
  - If $\lambda \leq 1-\alpha$,
    * If $\lambda+(1-\lambda)(1-F(\frac{H}{\alpha})) \geq \frac{1}{2}$, then $\tau^* = 0$. This is the case where capitalists still don't want any taxes, and they find a sizeable support from high-skilled workers and form a "coalition" to have zero taxes.
    * If $\lambda+(1-\lambda)(1-F(\frac{H}{\alpha})) < \frac{1}{2}$, then find h_1 which satisfies:

$$(1-\lambda)\int_0^{h_1} dF(h) = \frac{1}{2}$$

      We have: $\tau^* = g\left(1 - \frac{\alpha h_1}{H}\right) \in (0,1)$.
  - If $\lambda > 1-\alpha$, then find h_2 which satisfies:

$$(1-\lambda)\int_{h_2}^{\infty} dF(h) = \frac{1}{2}$$

    * If $g\left(1 - \frac{1-\alpha}{\lambda}\right) \leq \tau^*_{h_2}$, then $\tau^* = g\left(1 - \frac{1-\alpha}{\lambda}\right) \in (0,1)$.
    * If $g\left(1 - \frac{1-\alpha}{\lambda}\right) > \tau^*_{h_2}$, then $\tau^* = \tau^*_{h_2}$.

Finally, comparative statics. Note that in any case, the equilibrium tax rate does not depend on k. What about changes in *lambda*? Suppose we change λ to λ' with $\lambda' > \lambda$.

- If $\lambda > \frac{1}{2}$,
  - If $\lambda \leq 1 - \alpha$, τ^* begins at 0. If $\lambda' \leq 1 - \alpha$ it remains at zero, otherwise it increases to a strictly positive value.
  - If $\lambda > 1 - \alpha$, τ^* increases.
- If $\lambda < \frac{1}{2}$,
  - If $\lambda \leq 1 - \alpha$,
    * If $\lambda + (1-\lambda)(1-F(\frac{H}{\alpha})) \geq \frac{1}{2}$, τ^* starts at zero and remains at zero.
    * If $\lambda + (1-\lambda)(1-F(\frac{H}{\alpha})) < \frac{1}{2}$, then the effect of an increase in λ is ambiguous. We know that each worker prefers a lower tax rate than they did before now, but also the new median voter will have lower skill – which one prevails will depend on $F(.)$ and $v(.)$.
  - If $\lambda > 1 - \alpha$, once again, the effect on τ^* will depend on $F(.)$ and $v(.)$.

2. *Suppose now that agents vote over capital and labor income taxes, τ_k and τ_h, with corresponding costs $v(\tau_k)$ and $v(\tau_h)$, so that tax revenues are*

$$Tax = (\tau_k - v(\tau_k))\lambda rk + (\tau_h - v(\tau_h))(1-\lambda)w\int hdF(h)$$

Determine ideal tax rates for each agent. Suppose that $\lambda < 1/2$. Does a voting equilibrium exist? Explain. How does it change when λ increases? Explain why this would be different from the case with only one tax instrument?

The policy space right now is 2-dimensional: Indeed, it is the unit square $[0,1] \times [0,1]$, where, given a policy $(\tau_1, \tau_2) \in [0,1]^2$, τ_1 is the tax on capitalists and τ^2 is the tax on workers.

I'll not go over the calculations again – they're really similar to those in part 1. At the end of the day, it turns out each agent has single-peaked preferences on each dimension. A capitalist's bliss points are:

$$\tau_k^* = (0, g(1))$$

That is, a capitalist prefers zero taxes on capital income very high taxes on labor income (so high that the marginal distortion equals the gain from labor income tax). For a worker with human capital h, bliss points are:

$$\tau_h^* = (g(1), q(h))$$

where

$$q(h) := \begin{cases} 0, & \text{if } h \geq H \\ g(1 - \frac{h}{H}), & \text{if } h < H \end{cases}$$

That is, a worker always prefers high taxes on capital income. Her most preferred labor income tax depends on her skill level vis-a-vis the average skill level: she prefers lower taxes if she is more skilled, and may prefer zero labor taxes if she is sufficiently high-skilled.

The issue with this case is that: even though the preferences are single-peaked with respect to each dimension, because the policy space is multidimensional, the Median Voter Theorem may not hold and we may not have a Condorcet winner. To see this, assume that $\lambda < \frac{1}{2}$. Consider the most preferred labor income tax of the "median-skilled" worker h^m which satisfies:

$$(1 - \lambda) \int_{h^m}^{\infty} dF(h) = \frac{1}{2}$$

Realize that, for any τ_2, the policy $(g(1), \tau_2)$ beats any other policy (τ_1, τ_2) with $\tau_1 \neq g(1)$. Therefore, if we have a Condorcet winner, it must be of the form $(g(1), \tau_2)$. Also, the policy $(g(1), q(h^m))$ beats any policy $(g(1), \tau_2)$ with $\tau_2 \neq q(h^m)$. This means that if we have a Condorcet winner, it must be $(g(1), q(h^m))$ – the bliss point of the "median" worker.

But now, consider the policy $(g(1) - \varepsilon, q(h^m) - \varepsilon)$. For ε small enough, each capitalist and each worker with $h > h^m$ prefers this over $(g(1), q(h^m))$. This means we have a Condorcet cycle!

Now of course, if λ increases enough so that $\lambda > \frac{1}{2}$, this wouldn't be a problem – in that case, $(0, g(1))$ is the Condorcet winner.

3. *In this model with two taxes, now suppose that agents first vote over the capital income tax, and then taking the capital income tax as given, they vote on the labor income tax. Does a voting equilibrium exist? Explain. If an equilibrium exists, how does the equilibrium tax rate change when k increases? How does it change when λ increases?*

Voting in two stages save us from the non-existence of an equilibrium, because now we can consider two separate unidimensional policy spaces.

- If $\lambda \geq \frac{1}{2}$, $(\tau_1^*, \tau_2^*) = (0, g(1))$ (which was the Condorcet winner anyway) will still be implemented. The equilibrium tax rate does not change with k or λ.
- $\lambda < \frac{1}{2}$, the median voter is a worker and we'll have: $(\tau_1^*, \tau_2^*) = (g(1), q(h^m))$. The equilibrium tax rates are not affected by k, but now a change in λ affects the tax rate. If λ rises sufficiently high that the new median voter is a capitalist, we go to the case above. Otherwise, an effect of an increase in λ is ambiguous: it depends on $F(.)$ and $v(.)$, for reasons discussed in part 1.