# Introduction to Political Economy 14.770
# Problem Set 5 Solutions

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December 14, 2017

**Question 1:**

*This question builds on Esteban and Ray (2001). Consider an economy where there are two villages, village 1 and village 2, with a population $N_1$ and $N_2$ respectively. The regional government is deciding where to build a road that will go through one of this villages. Having the road built in village $j$ has a gross benefit*

$$\begin{cases} (1-\lambda)P + \lambda \frac{M}{N_i}, \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

*where $P > 0$ is a symmetric private benefit from having the road in your town and $M > 0$ is a public benefit to the town. The likelihood of having the road going through the town is a function of the relative effort that the village exerts. In particular, letting $a_{ij}$ denote the effort that individual $i$ in village $j$ exerts, the probability of getting the road through the town is*

$$\pi_j = \frac{A_j}{A_1 + A_2} = \frac{\sum_{i=1}^{N_j} a_{ij}}{\sum_{i=1}^{N_1} a_{ij} + \sum_{i=1}^{N_2} a_{ij}}$$

*The individual private cost of exerting effort is $v(a) = \frac{a^\beta}{\beta}$, where $\beta \geq 1$. Assume that individuals are risk neutral so that there expected utility is*

$$\pi_j \left[ (1-\lambda)P + \lambda \frac{M}{N_j} \right] - \frac{a_{ij}^\beta}{\beta}$$

1. *Solve for the symmetric Nash Equilibrium of the game above.*

   This is unfortunately a very ugly and messy calculation if you want to get the full, closed form solution. I'm posting solutions from an earlier year – see the supplementary document.

2. *How do the equilibrium effort levels (individual and aggregate) vary with $\lambda, P, M$, and $N_i$? Does the Olsonian conjecture hold in this case?*

   Even uglier! But there is a heuristic way of seeing this – which is more or less how Banerjee, Iyar and Somanathan (2007) approaches this.

   Recall that the first order condition for individual $i$ in group $j$ is:

$$\left(1 - \frac{A_j}{A}\right)\left[(1-\lambda)P + \lambda\frac{M}{N_j}\right] - (a_{ij})^{\beta-1} = 0$$

   Multiplying with $(N_j)^{\beta-1}$, realizing that $A_j = a_{ij}N_j$, and rearranging gives:

$$\left[(1-\lambda)P + \lambda\frac{M}{N_j}\right](N_j)^{\beta-1} = A_j \cdot A \cdot \left(1 - \frac{A_j}{A}\right)^{-1}$$

   Note that the right-hand side is increasing in $A_j$, and all the parameters we care about ($P, M, \lambda$ and $N_j$) are on the left hand-side. Consequently, if the left hand-side is increasing (decreasing) in a parameter, than equilibrium total effort for group $j$ is increasing (decreasing) in that parameter.

   This immediately yields that $A_j$ must be increasing in $P$ and $M$. The comparative statics for $\lambda$ and $N_j$ are more complicated, and in general they depend on other parameters (see the supplementary document). For instance, when $P = 0$, $A_j$ is increasing in $N_j$ iff $\beta > 2$. When $M = 0$, however, $A_j$ is always increasing in $N_j$.

**Question 2:**

*Consider the following extension to the Gentzkow and Shapiro (2006) model we saw in class, which adds with imperfect signals for the high types.*

*There are two type of newspapers: $\lambda$ fraction are high quality newspapers (denoted by $h$) and the remaining $1-\lambda$ fraction normal newspapers (denoted by $n$). The high quality newspapers receive a signal $s_h \in \{r, l\}$ of the true state of the world, $R, L$, and the signal is accurate with probability $\pi_h > 1/2$. The normal newspapers receive a signal $s_n \in \{r, l\}$ of the true state of the world and the signal is accurate with probability $\pi$, where $\pi_h \geq \pi > 1/2$.*

*Consumers want to learn the true state of the world a best as they can. They have a prior belief $\theta \in (1/2, \pi)$ of the true state of the world being $R$. They also get informative feedback with probability $\mu$.*

*Denote $X \in \{R, L, 0\}$ the feedback that consumers get. Define $\sigma_j(\widehat{j})$ to be the probability that a normal firm reports $\widehat{j}$, given that it gets signal $j$.*

*Normal firms want to maximize the posterior beliefs $\lambda(\widehat{j}, X)$ that agents have after getting a report $\widehat{j}$.*

1. *Suppose that a consumer gets a report $\widehat{r}$. What is the likelihood ratio of this coming form a high firm ($\Pr(\widehat{r}|h)/\Pr(\widehat{r}|n)$)? Is this increasing in $\theta$? How does it change with $\pi_h$?*

   Through simple Bayesian updating, one can derive:

$$\Pr(\widehat{r}|h) = \frac{\Pr(\widehat{r}, h)}{\Pr(h)} = \frac{\lambda\left(\theta\pi_h + (1-\theta)(1-\pi_h)\right)}{\lambda}$$

   and

$$\Pr(\widehat{r}|n) = \frac{\Pr(\widehat{r}, n)}{\Pr(n)} = \frac{(1-\lambda)\left[(\theta\pi + (1-\theta)(1-\pi))\,\sigma_r(\widehat{r}) + ((1-\theta)\pi + \theta(1-\pi))\,\sigma_l(\widehat{r})\right]}{(1-\lambda)}$$

   Therefore,

$$\frac{\Pr(\widehat{r}|h)}{\Pr(\widehat{r}|n)} = \frac{\theta\pi_h + (1-\theta)(1-\pi_h)}{(\theta\pi + (1-\theta)(1-\pi))\,\sigma_r(\widehat{r}) + ((1-\theta)\pi + \theta(1-\pi))\,\sigma_l(\widehat{r})}$$

   One can immediately observe that this is increasing in $\theta$ (ex ante likelihood of the true state) and $\pi_h$ (accuracy of the signal of high-quality newspaper). Formally,

$$\frac{\partial\left(\Pr(\widehat{r}|h)/\Pr(\widehat{r}|n)\right)}{\partial\theta} = \frac{(2\pi_h - 1)\Pr(\widehat{r}|n) - (\sigma_r(\widehat{r}) - \sigma_l(\widehat{r}))(2\pi - 1)\Pr(\widehat{r}|h)}{\Pr(\widehat{r}|n)^2} > 0$$

   and

$$\frac{\partial\left(\Pr(\widehat{r}|h)/\Pr(\widehat{r}|n)\right)}{\partial\pi} = \frac{2\theta - 1}{\Pr(\widehat{r}|n)} > 0$$

   What's the intuition?

- The likelihood ratio is increasing in $\theta$ because: as $\theta$ increases, probability that high type reports $\hat{r}$ increases faster than probability that normal type reports $\hat{r}$. This is because high quality newspapers have a more precise signal ($\pi_h \geq \pi$), and they're always "honest".
- The likelihood ratio is increasing in $\pi_h$ because: as $\pi_h$ increases, high quality newspaper observes a more accurate signal. Because the ex ante probability of $R$ is higher ($\theta > 1/2$), the more accurate signal is more likely to be $r$. Consequently, probability that high type reports $\hat{r}$ increases.

2. *Calculate the Posterior belief* $\lambda(\widehat{r}, 0) = \Pr(h|\widehat{r})$. *How does it change with* $\theta$ *and with* $\pi_h$?

   The posterior belief is:

$$\lambda(\widehat{r}, 0) = \frac{\lambda \Pr(\widehat{r}|h)}{\Pr(\widehat{r})} = \frac{\lambda \Pr(\widehat{r}|h)}{\lambda \Pr(\widehat{r}|h) + (1-\lambda)\Pr(\widehat{r}|n)} = \frac{1}{1 + \frac{(1-\lambda)\Pr(\widehat{r}|n)}{\lambda \Pr(\widehat{r}|h)}}$$

   Clearly, as $\Pr(\widehat{r}|h)/\Pr(\widehat{r}|n)$ increases with $\theta$ and $\pi_h$, the posterior is also increasing in the two parameters.

3. *What happens to bias in the case when there is no feedback? How does bias changes with* $\pi_h$? *What happens in the limit case when* $\pi_h \to \pi$? *(Note: Focus on the equilibrium which minimizes the slanting when firms receive signal* $\widehat{r}$*).*

   In the case with no feedback we must have $\lambda(\widehat{r}, 0) = \lambda(\widehat{l}, 0)$: This requires that the normal firm reports $\widehat{r}$ with probability $\theta\pi_h + (1-\theta)(1-\pi_h)$.

   Notice that if there was truthful reporting then the probability that the normal newspaper reports $\widehat{r}$ is $\theta\pi + (1-\theta)(1-\pi) < \theta\pi_h + (1-\theta)(1-\pi_h)$, which implies that there is bias in equilibrium. We will analyze one of the many possible equilibri– take $\sigma_r(\widehat{r}) = 1$. This implies that

$$\sigma_l(\widehat{r}) = \frac{(2\theta - 1)(\pi_h - \pi)}{\theta(1-\pi) + \pi(1-\theta)}$$

   Clearly the slanting increases with $\pi_h$ and in the limit when the noise of both the high quality and normal newspapers is equal (so that effectively each firm is "high" quality), there is no slanting.

   For the solutions to the later parts, see the supplementary document.