# Lecture 1: Fundamental Solution Concepts

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MIT

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# Introduction to 14.126

14.126 is an advanced PhD class on game theory and economic applications.

Target audience:

- Students in economics and related fields who want to use game theory in their research
  - IO, political economy, organizational economics, development, macro, ...
  - Computer science, political science, evolutionary biology, ...
- Students who want to do research in game theory

Correspondingly, class covers a mix of standard material (1970s-90s) and recent papers.

# Prerequisites

Basic game theory on the level of 14.12 or 14.122.

Some familiarity with real analysis and probability theory.

# Requirements

4 problem sets (60% of the grade)

Take-home final, 24 hours during exam week (40%)

# Outline: Alex's part

- Fundamental solution concepts and equilibrium refinements (1 week)
- Communication games: signaling and cheap talk (1 week)
- Repeated games (2 weeks)
- Reputation effects in games and markets (1 week)
- Bargaining (1 week)
- Social learning (guest lecture by Krishna Dasaratha on March 18th)

# Tentative Outline: Muhamet's part

- Supermodular games (2 weeks)
- Global games and potential games (2 weeks)
- Type spaces and belief hierarchies (1 week)
- Non-equilibrium learning in games (2 weeks)

# Today: Fundamental Models and Solution Concepts

- Strategic-form games
- Nash and correlated equilibrium
  - Existence
  - Upper hemi-continuity
- Rationalizability
- Bayesian games
- Nash eqm, correlated eqm, and rationalizability in Bayesian games

# Strategic-Form Games (of Complete Information)

A **strategic-form game** $G = (I, S, u)$ consists of

- a finite set of **players** $I = \{1, \dots, n\}$,
- a (pure) **strategy set** $S_i$ for each $i \in I$, where $S = \prod_i S_i$,
- a **vNM utility function** $u_i : S \to \mathbb{R}$ for each $i \in I$, where $u = (u_i)_i$.

# Mixed Strategies and Conjectures

A **mixed strategy** $\sigma_i$ is a probability distribution over $S_i$.

- Typically, $S_i$ is a compact metric space, $\Sigma_i = \Delta(S_i)$ is space of Borel distributions with topology of weak convergence (weak* topology).

In a **mixed strategy profile** $\sigma = (\sigma_1, \ldots, \sigma_n)$, the players randomize independently.

- Distribution on $S$ is the product distribution $\prod_i \sigma_i(s_i)$.

A **conjecture** (or **belief**) $\mu_{-i}$ for player $i$ is a probability distribution over $S_{-i}$.

- A player can conjecture that her opponents' strategies are correlated (even though, in any mixed strategy profile, players randomize independently).

Utility functions extend linearly to mixed strategies and conjectures: $u_i(\sigma)$, $u_i(s_i, \sigma_{-i})$, $u_i(s_i, \mu_{-i})$.

# Best Responses, Strict Dominance

Strategy $s_i$ is a **best response** to conjecture $\mu_{-i}$ (e.g., to opponents' pure strategies $s_{-i}$) if $u_i\left(s_i, \mu_{-i}\right) \geq u_i\left(s_i', \mu_{-i}\right)\ \forall s_i'$.

- $B_i\left(\mu_{-i}\right)$ =set of best responses to $\mu_{-i}$.
- Can analogously define when a mixed strategy is a best response.

$\sigma_i$ **strictly dominates** $s_i$ if $u_i\left(\sigma_i, s_{-i}\right) > u_i\left(s_i, s_{-i}\right)\ \forall s_{-i}$.

- $s_i$ is **strictly dominated** if some $\sigma_i$ strictly dominates it.
- A strategy can be strictly dominated even if it's not strictly dominated by any *pure* strategy. [Find an example.]

# Outline

- Strategic-form games
- **Nash and correlated equilibrium**
  - Existence
  - Upper hemi-continuity
- Rationalizability
- Bayesian games
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# Nash and Correlated Equilibrium

A **Nash equilibrium** (Nash, 1950) is a strategy profile $\sigma$ s.t. $u_i(\sigma) \geq u_i(s_i, \sigma_{-i}) \ \forall i, s_i$.

- i.e., $\sigma$ s.t. each $\sigma_i$ is a (mixed) best response to $\sigma_{-i}$.

A **(objective) correlated equilibrium** (Aumann, 1974) is a distribution $\mu \in \Delta(S)$ s.t. $u_i(\mu) \geq \sum_s \mu(s) u_i(d_i(s_i), s_{-i})$ $\forall i, d_i : S_i \to S_i$.

- i.e., $\mu$ s.t. each $s_i$ is optimal given information that $i$ takes $s_i$.

**Note:** the set of correlated equilibria is a convex polytope—subset of the convex set $\Delta(S)$ that satisfies a set of linear inequalities.

- In contrast, the set of Nash equilibria is not convex and not a polytope. This makes CE more convenient in some ways.

# Nash vs. Correlated

A CE is a NE in the "expanded game" where each player observes a private signal $r_i \in R_i$ before acting.

A "revelation principle" implies that it is without loss to consider the "canonical" signal space $R_i = S_i$ and focus on "obedient" equilibria where player $i$ takes $s_i = r_i$ after observing signal $s_i$.

In the special case where signals are independent (i.e., $\mu \in \prod_i \Delta(S_i)$), CE reduces to NE.

- CE generalizes NE, in the sense that any NE distribution $\mu \in \Delta(S)$ is a CE.

# Nash vs. Correlated (cntd.)

Interpretations of CE:

1. There is a mediator/principal who coordinates the players. Here, $r_i$ is the mediator's private message to player $i$. (With public messages, get convex combinations of NE.)
2. The game is played repeatedly and players observe private pre-play signals drawn from a fixed objective distribution. Then any steady state is a correlated equilibrium, and under some conditions play converges to a correlated equilibrium. (Without pre-play signals, a steady state is a NE. Convergence to NE is more subtle...)

# Nash Equilibrium Existence

## Theorem (Debreu, Fan, Glicksberg 1952)

*Suppose that each $S_i$ is a non-empty, convex, compact metric space and each $u_i : S \to \mathbb{R}$ is continuous in $s$ and quasi-concave in $s_i$. There there exists a pure-strategy Nash equilibrium.*

## Corollary (Nash 1950)

*Every finite game has a (possibly mixed) Nash equilibrium*

**Proof.** The game with $\tilde{S}_i = \Delta(S_i)$ satisfies the Debreu-Fan-Glicksberg conditions.

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Many important games (auctions, contests) have discontinuous payoffs, so the theorem is not always enough. A series of papers generalize it to allow certain discontinuities, including Dasgupta Maskin (ReStud 1986) and Reny (ECMA 1999). Reny (AnnRevEcon 2020) is an up-to-date survey.

# Nash Existence: Proof

Let $F : S \rightrightarrows S$ be the **best reply correspondence**:
$F_i(s) = B_i(s_{-i})$.
Note that $s$ is a NE iff $s \in F(s)$, i.e., $s$ is a fixed point of $F$.

Since $S$ is compact and utilities are continuous, Berge's maximum theorem implies that $F$ is non-empty and has a closed graph.

Moreover, since utilities are quasi-concave in own-actions, $F$ is also convex-valued.

By Kakutani's fixed point theorem, every non-empty, convex-valued correspondence with a closed graph on a non-empty, compact, convex subset of a metric space has a fixed point.

Hence, $F$ has a fixed point.

# Correlated Equilibrium Existence

Existence of a correlated equilibrium follows from existence of a Nash equilibrium.

However, since CE is defined by linear inequalities, it's natural to expect that existence can also be established directly by linear programming duality (without a topological fixed point theorem like Kakutani).

This was worked out by Hart and Schmeidler (MathOR 1989).

- We'll see a similar application of LP duality/minmax thm/separating hyperplane thm later in today's lecture.

# Upper Hemi-Continuity of NE

Similar arguments to Nash existence show that the set of NE is UHC (i.e., NE do not "disappear in the limit") in parameters that shift utilities continuously.

Suppose payoffs are $u_i : S \times X \to \mathbb{R}$, where $X$ is a space of parameters. Let $NE(x)$ and $PNE(x)$ be the mixed and pure NE correspondences.

### Theorem

*If $S$ and $X$ are compact metric spaces and each $u_i$ is continuous in $(s, x)$, then $NE(x)$ and $PNE(x)$ are compact-valued and upper hemi-continuous.*

**Proof.** Take $(x^m, \sigma^m) \to (x, \sigma)$ with $\sigma^m \in NE(x^m)\ \forall m$.
Then $u_i(\sigma^m; x^m) \geq u_i\left(s_i, \sigma^m_{-i}; x^m\right)\ \forall i, s_i, m$.
By continuity, inequalities are preserved in limit. So $\sigma \in NE(x)$.

$NE(x)$ is not lower hemi-continuous (i.e., NE can "appear in the limit"). [Find an example.]

# Outline

- Strategic-form games
- Nash and correlated equilibrium
  - Existence
  - Upper hemi-continuity
- **Rationalizability**
- Bayesian games
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# Dominance and Rationality

A strategy is “rational” if it is a best reply to some belief.

- I.e., maximizes subjective expected utility (Savage) under uncertainty about opponents’ strategies.

The next lemma shows that a strategy is rational in this sense iff it is not strictly dominated.

### Lemma (Pearce 1984)

*A strategy $s_i$ is never a best reply (to any conjecture $\mu_{-i}$) iff it is strictly dominated.*

**Proof.** Strictly dominated $\implies$ never a BR is immediate.

For converse, suppose $s_i$ is not strictly dominated. We will construct a conjecture $\mu_{-i}$ against which it’s a best reply.

# Not Strictly Dominated Implies Best Reply

- For any $\sigma_i$, let $\vec{u}_i(\sigma_i) = (u_i(\sigma_i, s_{-i}))_{s_{-i} \in S_{-i}}$.
- Let $U = \{\vec{u}_i(\sigma_i)\}_{\sigma_i \in \Sigma_i}$, the set of payoff vectors as a function of the opponents' strategies that $i$ can attain for some $\sigma_i$. Note that $U$ is non-empty and convex.
- Let $V = \left\{ v \in \mathbb{R}^{|S_{-i}|} : v > \vec{u}_i(s_i) \right\}$ be the set of payoff vectors that strictly dominate $s_i$. Note that $V$ is non-empty and convex.
- Since $s_i$ is not strictly dominated, $U$ and $V$ are disjoint. Hence, by the separating hyperplane theorem, there exists a non-zero vector $\lambda \in \mathbb{R}^{|S_{-i}|}$ such that $\lambda \cdot u \leq \lambda \cdot v$ $\forall u \in U, v \in V$. Moreover, since $V$ is unbounded in positive directions, we have $\lambda \geq 0$, so a normalized version of $\lambda$ is a conjecture $\mu_{-i}$.
- Finally, note that $\vec{u}_i(s_i) \in \mathrm{cl}(V)$, so $\mu_{-i} \cdot \vec{u}_i(s_i) \geq \mu_{-i} \cdot u$ $\forall u \in U$. Thus, $s_i \in B_i(\mu_{-i})$.

# Iterated Dominance and Rationalizability

Rationalizability is iterated rationality: a strategy is "rationalizable" if it is a best reply to a conjecture where others play best replies to conjectures where others play best replies to conjectures where. . .

By Pearce's lemma, a strategy is rationalizable iff it survives iterated deletion of strictly dominated strategies.

Formally, the set of **(correlated) rationalizable strategy profiles** is $S^\infty = \bigcap_{m=0}^{\infty} S^m$, where $S^0 = S$ and $S^m = \prod_i S_i^m$, where

$$S_i^m = B_i\left(\Delta\left(S_{-i}^{m-1}\right)\right) = S_i^{m-1} \setminus \left\{s_i : s_i \text{ is strictly dominated given } S_{-i}^{m-1}\right.$$

- (Equality by Pearce's lemma.)

# Fixed-Point Definition and Relation to Equilibrium

A set $R \subseteq S$ is **closed under rational behavior (CURB)** if $R_i \subseteq B_i\left(\Delta\left(R_{-i}\right)\right) \forall i$.

## Theorem

*Every CURB set is contained in $S^\infty$. Moreover, if $S$ is compact and utilities are continuous, then $S^\infty$ is CURB, and hence is the largest CURB set.*

**Proof.** If $R$ is CURB, then no strategy in $R$ is deleted at any round, so $R \subseteq S^\infty$.

Conversely, if $S$ is compact and utilities are continuous, then any $s_i \notin B_i\left(\Delta\left(S^\infty\right)\right)$ is also not in $B_i\left(\Delta\left(S^m\right)\right)$ for some $m$ and hence is deleted in round $m$, so $S^\infty$ is CURB.

**Note:** If $\mu$ is a correlated equilibrium then its support is CURB. Thus, only rationalizable strategies are played in a correlated equilibrium. [Find a counterexample to the converse.]

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# Bayesian Games

A **Bayesian game** $B = (I, \Theta, T, A, u, p)$ consists of

- a finite set of **players** $I = \{1, \ldots, n\}$,
- a set of **payoff-relevant parameters** $\Theta$,
- a **type space** $T_i$ for each $i$ (often, "possible realizations of $i$'s private information"),
- an **action set** $A_i$ for each $i$,
- a **belief** $p_i(\cdot|t_i) \in \Delta(\Theta \times T_{-i})$ for each $i, t_i$,
- a **vNM utility function** $u_i : A \times \Theta \to \mathbb{R}$ for each $i$.

An alternative definition dispenses with $\Theta$ and writes $u_i : A \times T \to \mathbb{R}$. This is formally a different model; whether the difference matters or not depends on the solution concept...

# Ex Ante and Interim Perspectives

A Bayesian game can be represented as a strategic-form game among the *players* (ex ante game) or the *types* (interim game).

**Ex ante game**: $G(B) = \left(I, A^T, (\mathbb{E}[u_i])_i\right)$

- Players: $I$
- Strategies: $s_i : T_i \to A_i$
- Payoffs: $U_i(s) = \mathbb{E}^s[u_i(a)] = \sum_{(\theta,t)} p_i(\theta, t)\, u_i(s(t), \theta)$

**Interim game:** $IG(B) = \left(\bigcup_i T_i, \prod_i \prod_{|T_i|} A_i, (\mathbb{E}[u_i|t_i])_{i,t_i}\right)$

- Players: $T$
- Strategies: $S_{t_i} = A_i$
- Payoffs: $U_{t_i}(a) = \mathbb{E}[u_i(a)\,|t_i] = \sum_{(\theta,t_{-i})} p_i(\theta, t_{-i}|t_i)\, u_i(a, \theta)$
- (Note: this notation implicitly assumes finite types. With infinite types, defining conditional expectation sometimes requires some care.)

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# Bayes Nash Equilibrium

It is natural to define a **Bayes Nash equilibrium** either as a NE in the ex ante game or a NE in the interim game.

If $p_i(t_i) > 0\ \forall i, t_i$, these definitions coincide, because $s_i : T_i \rightarrow A_i$ maximizes $\mathbb{E}^{(s_i, s_{-i})}[u_i(a)]$ iff $s_i(t_i)$ maximizes $\mathbb{E}[u_i(a_i, a_{-i})\,|t_i]$ $\forall t_i$.

If some types have 0 probability, the ex ante definition is more permissive, because it does not require optimizing conditional on 0-probability types.

- But the same "almost surely."

Existence and UHC of BNE (including in beliefs) follow from the corresponding results for complete info games.

# Common Prior Assumption

Our general formulation of a Bayesian game doesn't assume anything about how different types' beliefs relate to each other.

However, it is often assumed that beliefs are consistent with a **common prior**: there exists $P \in \Delta(\Theta \times T)$ such that each $p_i(\cdot | t_i)$ is derived by updating $p$ conditional on $t_i$ by Bayes' rule:

$$p_i(\theta, t_{-i} | t_i) = \frac{P(\theta, t_i, t_{-i})}{\sum_{(\theta', t'_{-i})} P(\theta', t_i, t'_{-i})} \qquad \forall i, t_i, \theta, t_{-i}.$$

# Common Prior: Pros and Cons

It is often convenient to work with a common prior:

- Simplifies model: specify $P$ rather than $\left(p_i\left(\cdot|t_i\right)\right)_{i,t_i}$.
- Takes the view that "differences in beliefs are only due to differences in information." This viewpoint is useful for focusing on effects of asymmetric information, which is one of the main reasons people study Bayesian games.
- With a common prior, can interpret $t_i$ as the realization of $i$'s private information.

However, common prior rules out "agreeing to disagree," which is also something we sometimes want to study.
E.g., trade due to fundamental disagreement in finance (Harrison Kreps 1979); "voice" and communication with heterogeneous priors (Banerjee Somanathan 2001; Sethi Yildiz 2016); optimism as a source of disagreement in bargaining (Yildiz 2003, 2004). See Morris (1996) for general discussion of the CPA.

Overall, CPA is the typical case, but non-CPA is also useful.

# Bayes Correlated Equilibrium

Traditionally, there have been various ways of defining correlated equilibrium in a Bayesian game (cf. Forges 1993).

The main modern notion is **Bayes correlated equilibria (BCE)** (Bergemann Morris 2016), which is a correlated equilibrium in the interim game with a common prior $P$: i.e., $\mu \in \Delta(\Theta \times T \times A)$ such that

1. $\text{marg}_{\Theta \times T} \mu = P$, and
2. $a_i$ maximizes $\sum_{(\theta, t_{-i}, a_{-i})} \mu(\theta, t_i, t_{-i}, a_i, a_{-i}) u_i(a_i', a_{-i}, \theta)$ $\forall i, t_i, a_i$.

2 interpretations:

1. Design perspective: each player $i$ knows $t_i$; mediator knows $(\theta, t)$; mediator privately recommends actions.
2. Outside observer perspective: observer knows each player $i$ knows $t_i$ but allows that players could learn more; ask what can happen in Bayes NE for some information structure?

# No Initial Information

A particularly tractable special case of BCE arises when the players have no initial information (Forges: "universal Bayesian solution").

Then a BCE is just $\mu \in \Delta(\Theta \times A)$ such that

1. $\text{marg}_{\Theta}\mu = P$, and
2. $a_i$ maximizes $\sum_{(\theta,a_{-i})} \mu(\theta, a_i, a_{-i}) u_i(a_i', a_{-i}, \theta)$ $\forall i, a_i$.

Since giving players more information just tightens obedience, this is the set of joint distributions on $\Theta \times A$ that arises in any BCE (or any Bayes NE for any info structure).

With 1 player and no initial info, BCE is the same as **information design** (Kamenica Gentzkow, 2011).

However, no initial info is not always realistic. E.g., may want to assume that competing firms know their own production costs, though they may not know demand or others' costs.

# Rationalizability in Bayesian Games

There are three natural definitions of rationalizability in Bayesian games (Dekel Fudenberg Morris, 2007):

**Ex ante rationalizability:** rationalizability in $G(B)$.

- Implicitly imposes common knowledge that players' beliefs about $\Theta \times T_{-i}$ are independent of their type and their beliefs about others' strategies (mappings from types to actions).

**Interim independent rationalizability:** rationalizability in $IG(B)$.

- Implicitly imposes CK that, conditional on types, other players' actions are independent of $\Theta$.

**Interim correlated rationalizability:** drop this independence.
$S_i^\infty[t_i] = \bigcap_{m=0}^\infty S_i^m[t_i]$, where $S_i^0[t_i] = A_i$ and $a_i \in S_i^m[t_i]$ iff $a_i$ maximizes $\sum_{(a_{-i},\theta,t_{-i})} u_i(a_i', a_{-i}, \theta)\,\mu(a_{-i}, \theta, t_{-i})$ for some $\mu$ s.t. $\mu(\cdot, \theta, t_{-i})$ is supported on $S_{-i}^{m-1}[t_{-i}]$ and $\mathrm{marg}_{\Theta \times T_{-i}}\mu = p(\cdot|t_i)$.