Learning : Adjustment with Persistent Noise

14.126 Game Theory

Muhamet Yildiz

Road Map

1. General Template
2. Kandori-Mailath-Rob Model
3. Summary of other important models

Main idea

- There is always small but positive probability of mutation.
- Then, a strict Nash equilibria may **not** be "stochastically stable."
- In, 2x2 symmetric games, risk-dominant equilibrium is selected.

Stochastic Adjustment

1. Consider a game.
2. Specify a state space Θ, e.g., the # of players playing a strategy.

1.

	A	B
A	2,2	0,0
B	0,0	1,1

2. Θ = {AA,AB,BA,BB}

Stochastic Adjustment, continued

3. Specify an adjustment dynamics, e.g., best-response dynamics, with a transition matrix *P*, where

   $P_{\theta,\xi} = \Pr(\theta \text{ at } t+1 | \xi \text{ at } t)$

ϕ = a probability distribution, a column vector.

3. $P =$

	AA	AB	BA	BB
AA	1	0	0	0
AB	0	0	1	0
BA	0	1	0	0
BB	0	0	0	1

Stochastic Adjustment, continued

4. Introduce a small noise: Consider P^ε, continuous in ε and $P^\varepsilon \to P$ as $\varepsilon \to 0$.

   Make sure that there exist a unique ϕ_ε^* s.t.

   $\phi_\varepsilon^* = P^\varepsilon \phi_\varepsilon^*$.

4. $P^\varepsilon =$

AA	AB	BA	BB
$(1-\varepsilon)^2$	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)\varepsilon$	ε^2
$(1-\varepsilon)\varepsilon$	ε^2	$(1-\varepsilon)^2$	$(1-\varepsilon)\varepsilon$
$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)^2$	ε^2	$(1-\varepsilon)\varepsilon$
ε^2	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)^2$

$\phi_\varepsilon^* = (1/4,1/4,1/4,1/4)^T$.

Stochastic Adjustment, continued

5. Verify that $\lim_{\varepsilon \to 0} \phi_\varepsilon^* = \phi^*$ exists; compute ϕ^*. (By continuity $\phi^* = P\phi^*$.)
6. Check that ϕ^* is a point mass, i.e.,

$$\phi^*(\theta^*) = 1$$

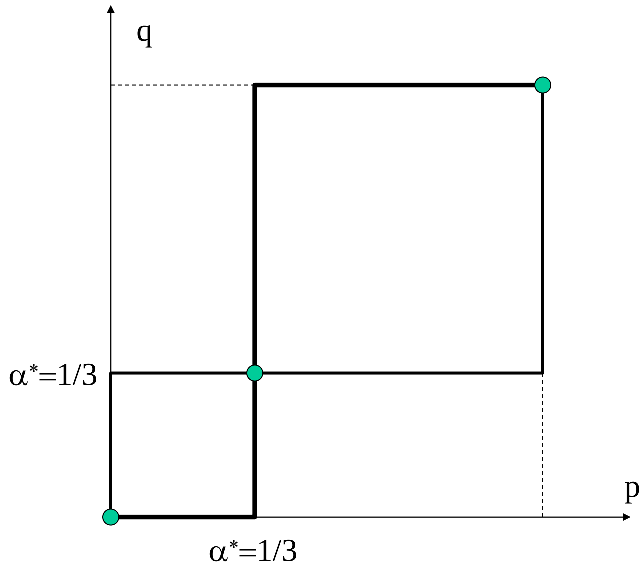
   for some θ^*.

The strategy profile at θ^* is called the *stochastically stable equilibrium*.

Stag Hunt

		
	(3,3)	(0,2)
	(2,0)	(2,2)

Best responses in Stag-Hunt game



Adjustment Process

- N = population size.
- State space: θ_t = # of players who play A at t.
- $u_A(\theta_t) = \frac{\theta_t}{N} u(A, A) + \frac{N-\theta_t}{N} u(A, B)$
- $\theta_{t+1} = \mathrm{P}(\theta_t)$, where $\mathrm{sign}(\mathrm{P}(\theta_t) - \theta_t) = \mathrm{sign}\big(u_A(\theta_t) - u_B(\theta_t)\big)$
- Example:

$$P(\theta_t) = BR(\theta_t) = \begin{cases} N \text{ if } u_A(\theta_t) > u_B(\theta_t) \\ \theta_t \text{ if } u_A(\theta_t) = u_B(\theta_t) \\ 0 \text{ if } u_A(\theta_t) < u_B(\theta_t) \end{cases}$$

Noise

- Independently, each agent with probability 2ε mutates, and plays either of the strategies with equal probabilities.

$$P^{\varepsilon} = \begin{bmatrix} (1-\varepsilon)^N & (1-\varepsilon)^N & \ldots & (1-\varepsilon)^N & \varepsilon^N & \ldots & \varepsilon^N \\ N(1-\varepsilon)^{N-1}\varepsilon & N(1-\varepsilon)^{N-1}\varepsilon & \ldots & N(1-\varepsilon)^{N-1}\varepsilon & N(1-\varepsilon)\varepsilon^{N-1} & \ldots & N(1-\varepsilon)\varepsilon^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N(1-\varepsilon)\varepsilon^{N-1} & N(1-\varepsilon)\varepsilon^{N-1} & \ldots & N(1-\varepsilon)\varepsilon^{N-1} & N(1-\varepsilon)^{N-1}\varepsilon & \ldots & N(1-\varepsilon)^{N-1}\varepsilon \\ \varepsilon^N & \varepsilon^N & \ldots & \varepsilon^N & (1-\varepsilon)^N & \ldots & (1-\varepsilon)^N \end{bmatrix}$$

- $\varphi^*(\varepsilon)$ = invariant distribution for P^{ε}.

Invariant Distribution

Define

- α^* by $u_A(\alpha^* N) = u_B(\alpha^* N)$
- $N^* = \lceil \alpha^* N \rceil < N/2$
- $D_A = \{\theta | \theta \geq N^*\}$; $D_B = \{\theta | \theta < N^*\}$;
- $q_{AB} = \Pr(\theta_{t+1} \in D_A | \theta_t \in D_B)$
- $q_{BA} = \Pr(\theta_{t+1} \in D_B | \theta_t \in D_A)$
- $p_A(\varepsilon) = \varphi^*_\varepsilon(D_A)$ and $p_A(\varepsilon) = \varphi^*_\varepsilon(D_B)$.

Invariant distribution, continued

- $$\begin{bmatrix} p_A(\varepsilon) \\ p_B(\varepsilon) \end{bmatrix} = \begin{bmatrix} 1 - q_{BA} & q_{AB} \\ q_{BA} & 1 - q_{AB} \end{bmatrix} \begin{bmatrix} p_A(\varepsilon) \\ p_B(\varepsilon) \end{bmatrix}$$
- $$\begin{bmatrix} -q_{BA} & q_{AB} \\ q_{BA} & -q_{AB} \end{bmatrix} \begin{bmatrix} p_A(\varepsilon) \\ p_B(\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
- $$\frac{p_A(\varepsilon)}{p_B(\varepsilon)} = \frac{q_{AB}}{q_{BA}}$$
- $$\frac{q_{AB}}{q_{BA}} = \frac{\binom{N}{N^*} \varepsilon^{N^*}(1-\varepsilon)^{N-N^*} + o(\varepsilon^{N^*+1})}{\binom{N}{N^*} \varepsilon^{N-N^*}(1-\varepsilon)^{N^*} + o(\varepsilon^{N-N^*+1})} \cong \left(\frac{\varepsilon}{1-\varepsilon}\right)^{2N^*-N}$$

Proposition

If N is large enough so that $N^* < (N+1)/2$, then limit φ* of invariant distributions puts a point mass on θ = N, corresponding to all players playing risk-dominant strategy A.

Other Models

- Freidlin & Wentzell (1984): invariant distribution of general processes.
- General result (Young, 93): invariant distribution is concentrated on the limit set ω that are "cheapest to be reached" from other ω'.
- Ellison (1993): local interaction games where mutations spread fast…