

Learning : Adjustment with Persistent Noise

14.126 Game Theory
Muhamet Yildiz

Road Map

1. General Template
2. Kandori-Mailath-Rob Model
3. Summary of other important models

Main idea

- There is always small but positive probability of mutation.
- Then, a strict Nash equilibria may **not** be “stochastically stable.”
- In, 2x2 symmetric games, risk-dominant equilibrium is selected.

Stochastic Adjustment

1. Consider a game.
2. Specify a state space Θ , e.g., the # of players playing a strategy.

1.

	A	B
A	2,2	0,0
B	0,0	1,1

2. $\Theta = \{AA, AB, BA, BB\}$

Stochastic Adjustment, continued

3. Specify an adjustment dynamics, e.g., best-response dynamics, with a transition matrix P , where

$$P_{\theta, \xi} = \Pr(\theta \text{ at } t+1 | \xi \text{ at } t)$$

ϕ = a probability distribution, a column vector.

3. AA AB BA BB

1	0	0	0	AA
0	0	1	0	AB
0	1	0	0	BA
0	0	0	1	BB

$P =$

Stochastic Adjustment, continued

4. Introduce a small noise: Consider P^ε , continuous in ε and $P^\varepsilon \rightarrow P$ as $\varepsilon \rightarrow 0$.

Make sure that there exist a unique ϕ_ε^* s.t.

$$\phi_\varepsilon^* = P^\varepsilon \phi_\varepsilon^*.$$

4. AA AB BA BB

$(1-\varepsilon)^2$	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)\varepsilon$	ε^2
$(1-\varepsilon)\varepsilon$	ε^2	$(1-\varepsilon)^2$	$(1-\varepsilon)\varepsilon$
$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)^2$	ε^2	$(1-\varepsilon)\varepsilon$
ε^2	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)^2$

$P^\varepsilon =$

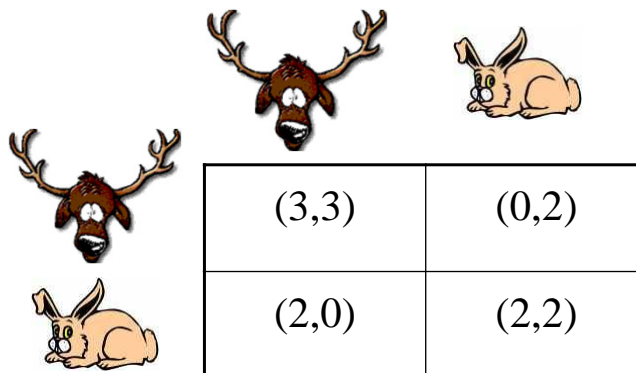
$$\phi_\varepsilon^* = (1/4, 1/4, 1/4, 1/4)^T.$$

Stochastic Adjustment, continued





5. Verify that $\lim_{\varepsilon \rightarrow 0} \phi_\varepsilon^* = \phi^*$ exists; compute ϕ^* . (By continuity $\phi^* = P\phi^*$.)
6. Check that ϕ^* is a point mass, i.e.,
$$\phi^*(\theta^*) = 1$$
for some θ^* .

The strategy profile at θ^* is called the *stochastically stable equilibrium*.

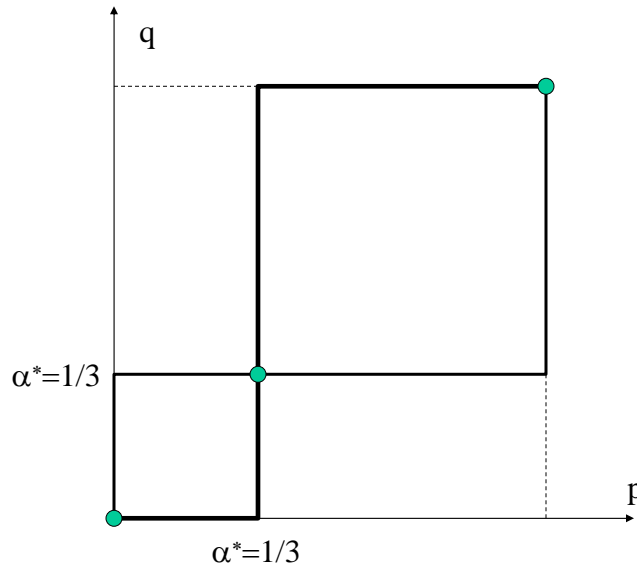
Stag Hunt



The Stag Hunt game matrix is shown with a 2x2 grid of payoffs. The columns represent the choices of the second player (Stag or Rabbit), and the rows represent the choices of the first player (Stag or Rabbit). The payoffs are (3,3) for (Stag, Stag), (0,2) for (Stag, Rabbit), (2,0) for (Rabbit, Stag), and (2,2) for (Rabbit, Rabbit). The icons for the animals are placed around the matrix: a stag icon above the top-left cell, a rabbit icon above the top-right cell, a stag icon to the left of the bottom-left cell, and a rabbit icon to the left of the bottom-right cell.

			
	(3,3)	(0,2)	
	(2,0)	(2,2)	

Best responses in Stag-Hunt game



Adjustment Process

- N = population size.
- State space: θ_t = # of players who play A at t .
- $u_A(\theta_t) = \frac{\theta_t}{N} u(A, A) + \frac{N-\theta_t}{N} u(A, B)$
- $\theta_{t+1} = P(\theta_t)$, where
 $\text{sign}(P(\theta_t) - \theta_t) = \text{sign}(u_A(\theta_t) - u_B(\theta_t))$
- Example:

$$P(\theta_t) = BR(\theta_t) = \begin{cases} N & \text{if } u_A(\theta_t) > u_B(\theta_t) \\ \theta_t & \text{if } u_A(\theta_t) = u_B(\theta_t) \\ 0 & \text{if } u_A(\theta_t) < u_B(\theta_t) \end{cases}$$

Noise

- Independently, each agent with probability 2ε mutates, and plays either of the strategies with equal probabilities.

$$P^\varepsilon = \begin{bmatrix} (1-\varepsilon)^N & (1-\varepsilon)^N & \dots & (1-\varepsilon)^N & \varepsilon^N & \dots & \varepsilon^N \\ N(1-\varepsilon)^{N-1}\varepsilon & N(1-\varepsilon)^{N-1}\varepsilon & \dots & N(1-\varepsilon)^{N-1}\varepsilon & N(1-\varepsilon)^{N-1}\varepsilon & \dots & N(1-\varepsilon)^{N-1}\varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N(1-\varepsilon)^{N-1}\varepsilon & N(1-\varepsilon)^{N-1}\varepsilon & \dots & N(1-\varepsilon)^{N-1}\varepsilon & N(1-\varepsilon)^{N-1}\varepsilon & \dots & N(1-\varepsilon)^{N-1}\varepsilon \\ \varepsilon^N & \varepsilon^N & \dots & \varepsilon^N & (1-\varepsilon)^N & \dots & (1-\varepsilon)^N \end{bmatrix}$$

- $\varphi^*(\varepsilon) =$ invariant distribution for P^ε .

Invariant Distribution

Define

- α^* by $u_A(\alpha^* N) = u_B(\alpha^* N)$
- $N^* = \lceil \alpha^* N \rceil < N/2$
- $D_A = \{\theta \mid \theta \geq N^*\}$; $D_B = \{\theta \mid \theta < N^*\}$;
- $q_{AB} = \Pr(\theta_{t+1} \in D_A \mid \theta_t \in D_B)$
- $q_{BA} = \Pr(\theta_{t+1} \in D_B \mid \theta_t \in D_A)$
- $p_A(\varepsilon) = \varphi_\varepsilon^*(D_A)$ and $p_B(\varepsilon) = \varphi_\varepsilon^*(D_B)$.

Invariant distribution, continued

- $$\begin{bmatrix} p_A(\varepsilon) \\ p_B(\varepsilon) \end{bmatrix} = \begin{bmatrix} 1 - q_{BA} & q_{AB} \\ q_{BA} & 1 - q_{AB} \end{bmatrix} \begin{bmatrix} p_A(\varepsilon) \\ p_B(\varepsilon) \end{bmatrix}$$
- $$\begin{bmatrix} -q_{BA} & q_{AB} \\ q_{BA} & -q_{AB} \end{bmatrix} \begin{bmatrix} p_A(\varepsilon) \\ p_B(\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
- $$\frac{p_A(\varepsilon)}{p_B(\varepsilon)} = \frac{q_{AB}}{q_{BA}}$$
- $$\frac{q_{AB}}{q_{BA}} = \frac{\binom{N}{N^*} \varepsilon^{N^*} (1 - \varepsilon)^{N - N^*} + o(\varepsilon^{N^* + 1})}{\binom{N}{N^*} \varepsilon^{N - N^*} (1 - \varepsilon)^{N^*} + o(\varepsilon^{N - N^* + 1})} \cong \left(\frac{\varepsilon}{1 - \varepsilon} \right)^{2N^* - N}$$

Proposition

If N is large enough so that $N^* < (N+1)/2$,
 then limit φ^* of invariant distributions puts
 a point mass on $\theta = N$, corresponding to all
 players playing risk-dominant strategy A.

Other Models

- Freidlin & Wentzell (1984): invariant distribution of general processes.
- General result (Young, 93): invariant distribution is concentrated on the limit set ω that are “cheapest to be reached” from other ω' .
- Ellison (1993): local interaction games where mutations spread fast...

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