# Potential Games

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# Potential Games

- In many applications, incentive to unilateral deviation may be as though they have a common payoff function
- ... as in common interest games
- ... although they have considerable amount of conflict
- Examples:
  - Prisoners' Dilemma
  - Cournot Oligopoly, Price competition
  - Most games in global games module
- Such games are called potential games
- ... and have many useful properties

# Outline

# Ordinal Potential Games

- Fix a complete information game $G = (N, S, u)$ with finite $N$
- **Definition:** $P : S \to \mathbb{R}$ is an *ordinal potential function* for $G$ if

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) > 0 \iff P(s_i, s_{-i}) - P(s_i', s_{-i}) > 0 \qquad (\forall i,$$

- $G$ is an *ordinal potential game* if it admits an ordinal potential function.
- Condition for a potential when applicable:

$$sign\left(\frac{\partial u_i}{\partial s_i}\right) = sign\left(\frac{\partial P}{\partial s_i}\right).$$

- **Example:** Cournot Oligopoly with inverse-demand function $\Pi$ and constant marginal cost:
- ... Payoff function:

$$u_i(q_1, \ldots, q_n) = q_i(\Pi(Q) - c)$$

- ... Ordinal Potential function:

$$P(q_1, \ldots, q_n) = q_1 \cdot \cdots \cdot q_n(\Pi(Q) - c)$$

# Pure Strategy Nash Equilibria

## Proposition

*Assume $P$ is an ordinal potential function for $G$. Then,*

1. *Pure NE $G$ coincide with pure NE of $(N, S, (P, \ldots, P))$.*
2. *Every $s \in \arg\max_{s \in S} P$ is a pure strategy Nash equilibrium of $G$.*
3. *If $S$ is compact and $u$ is continuous, then $G$ has a pure NE.*

- An *improvement path* is a sequence $s^0, s^1, \ldots$ with $i_0, i_1, \ldots$ s.t.
  - $s^m_{-i_m} = s^{m-1}_{-i_m}$ and $u_{i_m}(s^m) > u_{i_m}(s^{m-1})$ for each $m$.

## Proposition

*Let $P$ be an ordinal potential function for $G$. Then,*

1. *$P$ increases along every improvement path;*
2. *$\left|s^0, \ldots, s^m\right| \leq |S|$ when $S$ is finite;*
3. *every maximal improvement path terminates at a pure NE.*

# Potential Games

## Definitions

For $w = (w_1, \ldots, w_n) \in \mathbb{R}_+^N$, $P : S \to \mathbb{R}$ is a *w-potential function* for $G$ if

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = w_i \left(P(s_i, s_{-i}) - P(s_i', s_{-i})\right) \qquad (\forall i, s_i, s_i', s_{-i}).$$

$G$ is a *w-potential game* if it admits a $w$-potential function.
$P : S \to \mathbb{R}$ is a (exact) *potential function* for $G$ if

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i}) \qquad (\forall i, s_i, s_i', s_{-i}).$$

$G$ is a *potential game* if it admits an exact potential function.

- payoff function for a $w$-potential game: $u_i(s) = w_i P(s) + d_i(s_{-i})$
- Condition for a $w$-potential when applicable:

$$\frac{\partial u_i}{\partial s_i} = w_i \frac{\partial P}{\partial s_i} \qquad (\forall i).$$

- Unique up to a constant!

# Examples

- **Investment Game:**

| | $a$ | $b$ |
|---|---|---|
| $a$ | $\theta, \theta$ | $\theta - 1, 0$ |
| $b$ | $0, \theta - 1$ | $0, 0$ |

| | $a$ | $b$ |
|---|---|---|
| $a$ | $\theta$ | $0$ |
| $b$ | $0$ | $1 - \theta$ |

(1)

- **Prisoners' Dilemma**:

| | $C$ | $D$ |
|---|---|---|
| $C$ | $5, 5$ | $0, 6$ |
| $D$ | $6, 0$ | $1, 1$ |

| | $C$ | $D$ |
|---|---|---|
| $C$ | $0$ | $1$ |
| $D$ | $1$ | $2$ |

(2)

- **Modified Prisoners' Dilemma (Ordinal Potential)**:

| | $C$ | $D$ |
|---|---|---|
| $C$ | $5, 5$ | $0, 6$ |
| $D$ | $6, 0$ | $1, 2$ |

| | $C$ | $D$ |
|---|---|---|
| $C$ | $0$ | $1$ |
| $D$ | $1$ | $2$ |

(3)

- No Potential function for Modified Prisoners' Dilemma

# Potential Functions in Nice Games

## Proposition

*Assume: each $S_i$ is an interval and $u_i$ is twice continuously differentiable. Then, $G$ is a potential game iff*

$$\frac{\partial^2 u_i}{\partial s_i \partial s_j} = \frac{\partial^2 u_j}{\partial s_i \partial s_j} \qquad (\forall i, j \in N).$$

*A potential function (when exists):*

$$P(s) = \sum_{i \in N} \int_0^1 \frac{\partial u_i}{\partial s_i}(x(t))\, x_i'(t)\, dt$$

*where $x : [0,1] \to S$ is a (piecewise) continuously differentiable path with $x(0) = \hat{s}$ and $x(1) = s$ for some arbitrarily fixed $\hat{s} \in S$.*

# Examples of Nice Games:

- **Cournot Oligopoly:**

$$u_i(q_1,\ldots,q_n) = q_i(\Pi(Q) - c)$$
$$\frac{\partial^2 u_i}{\partial q_i \partial q_j} = \Pi'(Q) + q_i \Pi''(Q).$$

- ... a potential game iff demand is linear!
- ... Potential function for linear demand ($\Pi(Q) = a - bQ$):

$$P(q_1,\ldots,q_n) = a\sum_i q_i - b\sum_i q_i^2 - b\sum_{i<j} q_i q_j - \sum_i c_i(q_i).$$

- **Differentiated Price Competition:**

$$u_i(p_1,\ldots,p_n) = (p_i - c_i)\, Q_i(p_1,\ldots,p_n)$$
$$\frac{\partial^2 u_i}{\partial p_i \partial p_j} = \frac{\partial Q_i}{\partial p_j} + (p_i - c_i)\frac{\partial^2 Q_i}{\partial p_i \partial p_j}.$$

- ... a potential game iff demand functions are linear and symmetric!

# A general test for a potential

- A finite path: $\gamma = \left(s^0, \ldots, s^m\right)$ with $\iota = \left(i_0, \ldots, i_m\right)$ such that $s^k_{-i_k} = s^{k-1}_{-i_k}$ for all $k > 0$.
- A finite path is *closed* if $s^0 = s^m$;
- it is *simple* if $s^k \neq s^l$ for all other distinct pairs $(k, l)$.
- For any finite path $\gamma = \left(s^0, \ldots, s^m\right)$ define

$$I(\gamma) = \sum_{k=1}^{m} u_{i_k}\left(s^k\right) - u_{i_k}\left(s^{k-1}\right).$$

**Proposition**

*The following are equivalent.*

- *$G$ is a potential game.*
- *$I(\gamma) = 0$ for all finite closed paths.*
- *$I(\gamma) = 0$ for all finite, closed, simple paths of length 4.*

# Congestion Games

## Definition

A *congestion model* is a tupple $C = (N, M, S, c)$ where

- $N = \{1, \ldots, n\}$ is the set of players
- $M = \{1, \ldots, m\}$ is the set of facilities (or resources)
- $S = S_1 \times \cdots \times S_n$ is the set of strategy profiles where $S_i = 2^M \setminus \{\emptyset\}$
- $c_j \in \mathbb{R}^N$; $c_j(k)$ = payoff from using facility $j$ when $k$ players use.

A *congestion game*: $G = (N, S, u)$ where

$$\begin{aligned} u_i(s) &= \sum_{j \in s_i} c_j(k_j(s)) \\ k_j(s) &= \#\{i' \in N | j \in s_{i'} \qquad (\forall j \in M). \end{aligned}$$

# Congestion Games

**Theorem**

*Every congestion game is a potential game with*

$$P(s) = \sum_{j \in \cup_i s_i} \sum_{l=1}^{k_j(s)} c_j(l).$$

*Coversely, every finite potential game $G$ is isomoprhic to a congestion game.*

# Example

- Binary action game: $A_i = \{0, 1\}$,
- payoff from $a_i = 1$: $v(\alpha)$; $\alpha = \#$ players who take action 1
- and the payoff from action 0 is zero.
- ... as a congestion game: $c_0(k) = 0$ and $c_1(k) = v(k)$.
- The potential function:

$$P(s) = \sum_{l=1}^{\alpha(s)} v(l),$$

  where $\alpha(s) = \#$ of players who take action 1 under strategy profile $s$.
- A potential function for continuum of actions:

$$P(s) = \int_0^{\alpha(s)} v(x)\, dx.$$

- When $v$ is increasing, $P$ is maximized at $(a, \ldots, a)$ where $a_i = a$ is "risk-dominant", i.e., best response to the Laplacian belief!

# Correlated Equilibrium

## Theorem

*Let $G = (N, S, u)$ be a game with potential $P$ where*

- *each $S_i$ is convex, and*
- *$P$ is concave and continuously differentiable.*

*Then, a distribution $\mu$ on $S$ is a correlated equilibrium iff*

$$\mu\left(\arg\max P\right) = 1.$$

## Corollary

*If, in addition, $S$ is compact and $P$ is strictly concave, then there is a unique correlated equilibrium, which is a strict Nash equilibrium.*

# Bayesian Potential Games

- Consider a Bayesian game $\mathcal{B} = (N, \Theta, T, A, u, p)$ where $u_i : A \times \Theta \times T \to \mathbb{R}$
- $\mathcal{B} = (N, \Theta, T, A, u, p)$ is a *potential game* if there is $P : A \times \Theta \times T \to \mathbb{R}$ s.t.

$$u_i(a_i, a_{-i}, \theta, t) - u_i(a'_i, a_{-i}, \theta, t) = P(a_i, a_{-i}, \theta, t) - P(a'_i, a_{-i}, \theta, t).$$

## Proposition

*Let $\mathcal{B} = (N, \Theta, T, A, u, p)$ be a Bayesian potential game with potential function $P : A \times \Theta \times T \to \mathbb{R}$ and a common prior $p$. Then, the ex-ante game $G(\mathcal{B})$ is a potential game with potential function $\mathcal{P} : A^T \to \mathbb{R}$,*

$$\mathcal{P}(s) = E[P(s, \theta, t)].$$

*If in addition $p(t_i) > 0$ for each $t_i \in T_i$, then each $s \in \arg\max \mathcal{P}(s)$ is a BNE.*

# Bayesian Potential Games—Counterexample

- Consider the state dependent common-interest game:

$\theta_1$

| | $a$ | $b$ |
|---|---|---|
| $a$ | 1 | 0 |
| $b$ | 0 | 1 |

$\theta_2$

| | $a$ | $b$ |
|---|---|---|
| $a$ | 0 | 1 |
| $b$ | 1 | 0 |

- Player $i$ assigns probability 1 on $\theta_i$.
- Ex-ante game is not an ordinal potential game!
- Suppose now Player 1 knows $\theta$ and Player 2 assigns equal probabilities on each $\theta$.
- The interim game is not an ordinal potential game.

# Global Supermodular Potential Games

- Bayesian game $G^\sigma$, indexed by the shock size $\sigma$:
  - $N = \{1, \ldots, n\}$;
  - $\Theta \subseteq \mathbb{R}$;
  - $A_i$ is a countable union of closed intervals within $[0, 1]$ where $0, 1 \in A_i$;
  - $u_i : A \times \Theta \to \mathbb{R}$ is continuous with bounded derivatives;
  - each player observes a signal

    $$x_i = \theta + \sigma\varepsilon_i$$

    where $(\theta, \varepsilon_1, \ldots, \varepsilon_n)$ are stochastically independent with atomless densities, $\theta$ has full support, and $\varepsilon_1, \ldots, \varepsilon_n$ are bounded.
- FMP Assumptions: Strategic complementarity, Dominance regions, state monotonicity
- Assume: $G^\sigma$ is a Bayesian potential game with potential function $P : A \times \Theta \to \mathbb{R}$

# Noise-Independent Selection

## Theorem

*There exists a weakly increasing strategy profile* $s^*$ *such that, for every* $x = (x_1, x_1, \ldots, x_1)$ *at which* $s^*$ *is continuous,*

1. *for every* $\epsilon > 0$*, there exists* $\bar{\sigma} > 0$ *such that*

$$S_i^{\infty}\left[x_i | G^{\sigma}\right] \subset \left(s^*\left(x_i\right) - \epsilon, s^*\left(x_i\right) + \epsilon\right) \qquad \left(\forall \sigma < \bar{\sigma}\right);$$

2. $s^*\left(x\right)$ *is a Nash equilibrium of the complete information game in which it is common knowledge that* $\theta = x_1$*;*
3. *at each* $\theta$ *where each* $u_i\left(a_i, a_{-i}, \theta\right)$ *is quasi-concave in* $a_i$*,*

$$s^*\left(\theta, \ldots, \theta\right) \in \arg\max_{a \in A} P\left(a, \theta\right).$$