Supermodular Games

Muhamet Yildiz
14.126 Game Theory
MIT Economics Department

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Outline

Complementarity

- **Constraints:** Activities are complementary if doing one enables doing the other.
  - i.e. the domain is a lattice.
- **Payoffs:** Activities are complementary if doing one makes it weakly more profitable to do the other.
  - i.e. payoffs are supermodular.
- **Main Lesson:** When a and b are complementary, a higher input a leads to a higher output b
  - in optimization problems
  - and in strategic environments.

Example—Diamond's Search Model

- A continuum of players, i.
- Each exerts effort $a_i \in [0,1]$ and obtains payoff

$$U_i(a) = \theta a_i g(\bar{a}_{-i}) - a_i^2/2.$$

where

  - θ is value of a match,
  - $\bar{a}_{-i}$ is the average search by others,
  - $a_i g(\bar{a}_{-i})$ is probability of match where $g : [0,1] \to [0,1]$ is increasing, continuous.

- **Strategic complementarity:** $\partial U_i / \partial a_i$ is increasing in $\bar{a}_{-i}$.
- leads to an increasing best-response function:

$$B_i(a_{-i}) = \theta g(\bar{a}_{-i}).$$

- Complementarity between a_i and θ:

$$\partial^2 U_i / \partial a_i \partial \theta = g(\bar{a}_{-i}) \geq 0.$$

Lattices

Definition

A partially-ordered set $(X, \geq)$ is lattice if for all $x, y \in X$

$$\begin{aligned} x \vee y &\equiv \inf \{z \in X | z \geq x, z \geq y\} \in X \\ x \wedge y &\equiv \sup \{z \in X | x \geq z, y \geq z\} \in X. \end{aligned}$$

Example

$X = \mathbb{R}^n$ with the usual coordinate-wise order:

$$(x_1, ..., x_n) \geq (y_1, ..., y_n) \iff x_i \geq y_i \quad \forall i.$$

$(\mathbb{R}^n, \geq)$ is a lattice with

$$\begin{aligned} x \vee y &= (\max\{x_1, y_1\}, ..., \max\{x_n, y_n\}) \\ x \wedge y &= (\min\{x_1, y_1\}, ..., \min\{x_n, y_n\}). \end{aligned}$$

Complete Lattices

Definition

A lattice $(X, \geq)$ is said to be *complete* if for every $S \subseteq X$, a greatest lower bound $\inf(S)$ and a least upper bound $\sup(S)$ exist in X, where $\inf(\varnothing) = \sup(X)$ and $\sup(\varnothing) = \inf(X)$.

Example

- $X = 2^S$ and $A \geq B \iff A \supseteq B$.
- $A \vee B = A \cup B \in X$ and $A \wedge B = A \cap B \in X$.
- Therefore, $(X, \supseteq)$ is a lattice.
- Complete: $\vee_\alpha A_\alpha = \cup_\alpha A_\alpha \in 2^S$ and $\wedge_\alpha A_\alpha = \cap_\alpha A_\alpha \in 2^S$.

Strong Set Order and Sublattices

Definition (Strong Set Order)

Given any lattice $(X, \geq)$, for any $A, B \subseteq X$, write $A \geq B$ iff

$$x \vee y \in A, x \wedge y \in B \qquad (\forall x \in A, y \in B).$$

Example:

$$\begin{aligned} \{1,2,3,4\} &\geq \{0,1,2,3\} \\ &\not\geq \{-0.5, 0.5, 1.5, 2.5\} \end{aligned}$$

Definition

$S \subseteq X$ is sublattice if for any $x, y \in S$,

$$x \vee y \in S \quad \text{and} \quad x \wedge y \in S,$$

i.e., $S \geq S$.

Supermodular Functions

Definition

$f : T \longrightarrow X$ is isotone (or *weakly increasing*) if

$$t \geq t' \Rightarrow f(t) \geq f(t').$$

Definition

$f : X \longrightarrow \mathbb{R}$ is supermodular if for all $x, y \in X$

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

f is submodular if $-f$ is supermodular.

- When $X = X_1 \times X_2$, ordered coordinate-wise,

$$f(x_1, y_2) - f(x_1, x_2) \geq f(y_1, y_2) - f(y_1, x_2).$$

- For smooth functions on $\mathbb{R}^2$: $\frac{\partial^2 f}{\partial_{x_1} \partial_{x_2}} \geq 0$

Supermodularity on Product Spaces

- For lattices $(X_1, \geq_1), \ldots, (X_n, \geq_n)$, let $X = X_1 \times \cdots \times X_n$ and

$$(x_1, \ldots, x_n) \geq (y_1, \ldots, y_n) \iff x_i \geq_i y_i \qquad \forall i.$$

- For $f : X \to \mathbb{R}$, define $f(\cdot|x_{-ij}) : X_i \times X_j \to \mathbb{R}$ by $f(x_i, x_j|x_{-ij}) = f(x_i, x_j, x_{-ij})$.
- **Definition:** $f : X \to \mathbb{R}$ has increasing differences if

$$\left[x_i \geq x_i' \text{ and } x_j \geq x_j'\right] \Rightarrow \begin{array}{c} f(x_i, x_j, x_{-ij}) - f(x_i', x_j, x_{-ij}) \\ \geq f(x_i, x_j', x_{-ij}) - f\left(x_i', x_j', x_{-ij}\right) \end{array}$$

- When $X = \mathbb{R}^n$, this is called pair-wise supermodularity.
- **Lemma:** If f has increasing differences and $x_j \geq y_j$ for each j, $f(x_i, x_{-i}) - f(y_i, x_{-i}) \geq f(x_i, y_{-i}) - f(y_i, y_{-i})$.
- **Theorem:** f is supermodular if and only if
  1. f has increasing differences and
  2. f is supermodular within X_i for each i.

Outline

Monotonicity Theorem

Theorem (Topkis's Monotonicity Theorem)

For any lattices $(X, \geq)$ *and* $(\Pi, \geq)$, *let* $u : X \times \Pi \to \mathbb{R}$ *be a supermodular function (with coordinate-wise order) and define*

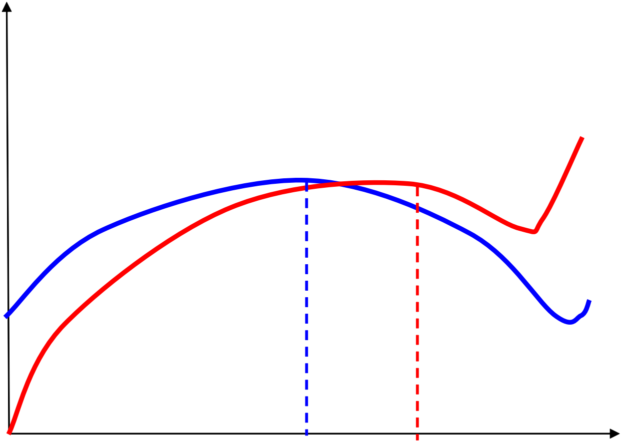
$$B(\pi) = \arg \max_{x \in D(\pi)} u(x, \pi).$$

If $\pi \geq \pi'$ *and* $D(\pi) \geq D(\pi')$, *then* $B(\pi) \geq B(\pi')$.

Corollary

For any fixed π, *if* $u(\cdot, \pi) : X \to \mathbb{R}$ *is supermodular and* $D(\pi)$ *is a sublattice of* X, *then* $B(\pi)$ *is a sublattice of* X.

Monotonicity Theorem—Illustration



Monotonicity Theorem—Proof

1. Take $\pi \geq \pi'$, $D(\pi) \geq D(\pi')$, $x \in B(\pi)$ and $x' \in B(\pi')$.
2. Need to show: $x \vee x' \in B(\pi)$ and $x \wedge x' \in B(\pi')$.
3. Since $x \in B(\pi) \subseteq D(\pi)$, $x \in D(\pi)$. Similarly, $x' \in D(\pi')$.
4. Since $D(\pi) \geq D(\pi')$, by 3, $x \vee x' \in D(\pi)$ and $x \wedge x' \in D(\pi')$.
5. Suffices: $u(x \vee x', \pi) = u(x, \pi)$ and $u(x \wedge x', \pi') = u(x', \pi')$.
6. By 1 and 4,

$$\begin{aligned} u(x \vee x', \pi) &\leq u(x, \pi) \\ u(x \wedge x', \pi') &\leq u(x', \pi'). \end{aligned}$$

7. If either inequality is strict, supermodularity fails:

$$u(x \vee x', \pi) + u(x \vee x', \pi') < u(x, \pi) + u(x', \pi').$$

Application—Pricing

- Under demand function $D(p, \theta)$ and marginal cost c, a monopolist sets a price

$$p^*(\theta, c) = \arg\max_{p \geq c} (p - c) D(p, \theta)$$

where $\theta \in \Theta$ is a demand parameter.

- Observe:

$$p^*(\theta, c) = \arg\max_{p \geq c} \log(p - c) + \log D(p, \theta).$$

- p^* is isotone in c because of supermodularity w.r.t. (p, c).
- p^* is isotone in θ whenever $\log D(p, \theta)$ is supermodular
- ... whenever the price elasticity of demand

$$-\frac{\partial \log D}{\partial \log p}$$

is weakly decreasing in θ.

Application—Pricing under Demand Uncertainty

- Monopolist does not know θ and has belief π about θ;

$$\tilde{D}(p,\pi) = E_\pi[D(p,\theta)].$$

- Assume D is isotone in θ and supermodular; $c = 0$.
- Monopolist sets price

$$p^*(\pi) = \arg\max_{p \geq 0} p\tilde{D}(p,\pi).$$

- Optimal price is isotone in monopolist's belief:

$$\pi \geq_{FOSD} \pi' \Longrightarrow p^*(\pi) \geq p^*(\pi').$$

- **Proof:** Apply Monotonicity Theorem:
  1. $(\Delta(\Theta), \geq_{FOSD})$ is a lattice (Exercise).
  2. Since D is increasing in θ, $\tilde{D}$ is isotone in π, and
  3. since D is supermodular, so is $\tilde{D}$ (prove these);
  4. p is trivially isotone and supermodular.
  5. Hence, $p\tilde{D}(p,\pi)$ is supermodular (Exercise).

Extensions and Generalizations

Definitions

A function $f : X \to \mathbb{R}$ on a lattice is said to be *quasi-supermodular* if for any $x, y \in X$,

$$\begin{aligned} f(x) &\geq f(x \wedge y) \Rightarrow f(x \vee y) \geq f(y) \\ f(x) &> f(x \wedge y) \Rightarrow f(x \vee y) > f(y). \end{aligned}$$

A function $f : X \times \Pi \to \mathbb{R}$ is said to have *single crossing property in* (x, π) if for any $x > x'$ and $\pi > \pi'$

$$\begin{aligned} f(x, \pi') &\geq f(x', \pi') \Rightarrow f(x, \pi) \geq f(x', \pi) \\ f(x, \pi') &> f(x', \pi') \Rightarrow f(x, \pi) > f(x', \pi) \end{aligned}$$

Theorem (Milgrom and Shannon)

Let $f : X \times \Pi \to \mathbb{R}$, where X is a lattice and Π is a partially ordered set. Then, for all $(\pi, D), (\pi', D') \in \Pi \times 2^X$,

Expected Utility Theory

Definition

A function $f : X \to \mathbb{R}$ is said to be *log-supermodular* if $\log f$ is supermodular.

Theorem (Athey)

Consider an expected utility maximizer with utility function $u : X \times \Pi \times \Theta \to \mathbb{R}$ *and density* $f : \Theta \times \Pi \to \mathbb{R}$. *If both* u *and* f *are log-supermodular, then*

$$B(\pi) = \arg\max_{x \in X} \int \Big(u(x, \pi, \theta) f(\theta, \pi) \, d\theta$$

is isotone.

Monotonicity under Completeness and Continuity

- Consider a complete lattice $(X, \geq)$ and $u : X \to \mathbb{R}$.
- **Definition:** u is continuous if

$$\lim u(x_n) = u\left(\sup x_n\right) \text{ and } \lim u(y_n) = u\left(\inf y_n\right)$$

for any (x_n) with $x_n \geq x_{n-1}$ and (y_n) with $y_n \geq y_{n+1}$ for all n.

Theorem

Let

- $(X, \geq)$ *and* $(\Pi, \geq)$ *be complete lattices,*
- $u : X \times \Pi \to \mathbb{R}$ *be continuous, supermodular w.r.t.* x *and has increasing differences.*

Then,

$$B(\pi) = \arg\max_{x \in X} u(x, \pi)$$

is a complete lattice and isotone; $\bar{B}(\pi) \equiv \max B(\pi) \in B(\pi)$ *and* $B(\pi) \equiv \min B(\pi) \in B(\pi)$ *exist and isotone.*

Outline

Supermodular Games—Formulation

Definition

A game (N, S, u) is supermodular if for each player $i \in N$,

- strategy space $(S_i, \geq_i)$ is a complete lattice for some order $\geq_i$,and
- u_i is continuous, supermodular in s_i and has increasing differences:

$$u_i(s \vee s') + u_i(s \wedge s') \geq u_i(s) + u_i(s') \qquad (\forall s_i, s_i', \forall s_{-i} \geq s_{-i}').$$

Since S is a complete lattice, $\underline{s} = \min S$ and $\overline{s} = \max S$ exist.

Linear Oligopoly Models

- Differentiated Bertrand Competition: n firms; each firm sets price p_i and gets profit

$$u_i(p) = (p_i - c_i)\, Q_i(p) = (p_i - c_i) \quad \theta - a_i p_i + \sum_{j \neq i} b_{ij} p_j \Bigg) \Bigg($$

- ... supermodular (whenever b_{ij} are all non-negative).
- Cournot Duopoly: $n = 2$ firms; each firm sets quantity q_i and gets profit

$$u_i(p) = q_i(\theta - Q - c_i) \text{ where } Q = q_1 + \cdots + q_n.$$

- ... supermodular when q_2 is ordered in the reverse order.
- Cournot Oligopoly: $n > 2$ firms.
- submodular...
- ... and cannot be made supermodular.

Fundamental Lemmas

Lemma

For any supermodular game, any $i \in N$,

1. *for every $s_{-i} \in S_{-i}$,*

$$B_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

   is a complete lattice;

2. *for every s, $\bar{B}_i(s) \equiv \max B_i(s_{-i}) \in B_i(s_{-i})$ and $\underline{B}_i(s) \equiv \min B_i(s_{-i}) \in B_i(s_{-i})$, and*
3. *$\bar{B}_i$ and $\underline{B}_i$ are isotone, i.e., $\bar{B}_i(s) \geq \bar{B}_i(s')$ and $\underline{B}_i(s) \geq \underline{B}_i(s')$ whenever $s \geq s'$.*

Lemma

Every s_i with $s_i \not\geq \underline{B}_i(\underline{s})$ is strictly dominated by $s_i \vee \underline{B}_i(\underline{s})$, where $\underline{s} = \min S$.

Rationalizability and Equilibrium

Theorem

For any supermodular game,

1. $\bar{z} \equiv \lim_k \bar{B}^k(\bar{s}) \equiv \inf_k \bar{B}^k(\bar{s})$ *and* $\underline{z} \equiv \lim_k \underline{B}^k(\underline{s}) \equiv \sup_k \underline{B}^k(\underline{s})$ *exists, where* $\bar{s} = \sup S$ *and* $\underline{s} = \inf S$;
2. *for every rationalizable strategy profile* s.

$$\bar{z} \geq s \geq \underline{z},$$

3. *and* $\bar{z}$ *and* $\underline{z}$ *are (pure strategy) Nash equilibria.*

Corollary

A supermodular game is dominance solvable if and only if there exists a unique Nash equilibrium in pure strategies.

A Partnership Game

- **Players:** an employer, who provides capital K,
- and a worker, who provides labor L.
- They share the output: $K^{\alpha}L^{\beta}$ for some $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$.
- The **utility functions**: $K^{\alpha}L^{\beta}/2 - K$ and $K^{\alpha}L^{\beta}/2 - L$.

Comparative Statics

Theorem

- *A family of supermodular games* $G^t = (N, S, U(\cdot; t))$.
- *For all* i, s_{-i}, $U_i(s_i, s_{-i}; t)$ *is supermodular in* (s_i, t).
- *Write* $\overline{z}(t)$ *and* $\overline{z}(t)$ *for the extremal equilibria at* t.
- *Then,* $\overline{z}(t)$ *and* $\underline{z}(t)$ *are isotone.*

Monotone Supermodular Games

Definition

A monotone supermodular game is a Bayesian game $\mathcal{B} = (N, A, \Theta, T, u, p)$ with

- each A_i is a compact sublattice of $\mathbb{R}^K$;
- $\Theta \times T$ is a measurable subset of $\mathbb{R}^M$;
- u_i is such that
  - $u_i(a, \cdot) : \Theta \to R$ is measurable,
  - $u_i(\cdot, \theta) : A \to R$ is continuous, bounded by an integrable function, supermodular in a_i and has increasing differences,
  - u_i has increasing differences in (a_i, θ), and
- $p(\cdot|t_i)$ is a weakly increasing function of t_i in the sense of first-order stochastic dominance.

Monotone Supermodular Games—Main Result

Theorem

Any monotone supermodular game has Bayesian Nash equilibria s^* and s^{**} in pure strategies such that

1. *for any t_i and any ICR action $a_i \in S_i^\infty[t_i]$ for t_i,*

$$s_i^*(t_i) \geq a_i \geq s_i^{**}(t_i),$$

2. *for any Bayesian Nash equilibrium s,*

$$s^*(t) \geq s(t) \geq s^{**}(t) \qquad (\forall t \in T),$$

3. *$s_i^*(t_i)$ and $s_i^{**}(t_i)$ are weakly increasing in t_i.*