Solutions for Problem Set 1
Game Theory for Strategic Advantage (15.025)
Spring 2015

1. Hotelling's Location Game. **The key to approaching this problem is to remember that the notion of dominance is all about my own payoff. For dominance, it never matters whether I get more or less than my opponent. Notice that all numbers in this answer are my own payoffs.**

   (a) Position 1 is strictly dominated by 2 (in fact, by many other strategies). This is because *my own payoff is higher when I choose 1 than when I choose 2, and this is true for any strategy of my opponent.* To see this, just consider each possible position of the other firm. Let's use the notation that the first entry is our firm and the second entry is the other firm. So, for example,

$$\begin{aligned} 90 &= u(2,1) > u(1,1) = 50 \\ 50 &= u(2,2) > u(1,2) = 10 \\ 20 &= u(2,3) > u(1,3) = 15 \\ 25 &= u(2,4) > u(1,4) = 20 \\ 30 &= u(2,5) > u(1,5) = 25 \end{aligned}$$

   and so on. A similar reasoning applies to strategies $3,4,5,6,7$. Strategies 8,9,10 do not dominate 1. To see this, suppose the other firm chooses position 7. Then $u(1,7) = 35$ and $u(8,7) = 30$, which means choosing 8 is not better than choosing 1 for all possible strategies of my opponent.

   (b) Strategy 1 is dominated by 2 (although this time only weakly). To see this for 1 and 2, let x and y (with $x \geq y$) be the positions of the other firms. Start with the easy cases. Then do the hard cases.

   Consider the cases where $x > y > 2$. In these cases, x is irrelevant when comparing choosing 1 or 2 . Whether our firm chooses 1 or 2, it wins all the consumers at or to the left of its position and half the consumers between herself and y, so moving to the right (from 1 to 2) increases its market share. The same analysis works, more or less, for the case $x = y > 2$. Again, whether our firm chooses 1 or 2, it wins all the consumers at or to the left of her position. Now, the fraction of consumers she wins between itself and y is not cleanly a half, but it is still a fraction less than one. So, again, moving to the right (from 1 to 2) increases market share.

   This only leaves four cases to check: $x = 3y = 2, x = y = 2, x = y = 1$, and $x = 2y = 1$. It is easy to check each of these:

$$\begin{aligned} u(2,2,3) &= 10 = 10 = u(1,2,3) \\ u(2,2,2) &= 33.33 > 10 = u(1,2,2) \\ u(2,1,1) &= 90 > 33.33 = u(1,1,1) \\ u(2,2,1) &= 45 > 10 = u(1,2,1) \end{aligned}$$

The same reasoning applies to strategy 3 – it weakly dominates strategy 1.

(c) In this three firm case, even after eliminating 1 and 10, position 3 does not dominate position 2. To see this, consider the case where the other firms choose 4 and 3. Then:

$$u(2,4,3) = 20 > 15 = u(3,4,3).$$

Strategy 2 is also not dominated by 4. Consider the other two firms choosing 5 and 3. A similar argument rules out any other strategy (for example 4) dominating 2.

2. Penalty Shots. **This problem is not asking a prediction for the game. It asks "what must a rational player be thinking in order to justify the choice of a given action?".**

(a) No strategy is dominated. For player 2, each strategy (l, m, r) is the best response to the corresponding strategy (L, M, R) by player 1. For player 1, M is not dominated by L and R because, if player 2 goes r, it is a better to choose M then R (and likewise for L).

(b) From Player 2's perspective, let p_M be the probability that player 1 chooses M, and p_L the probability that player 1 chooses L. Then the probability of player 1 choosing R is $p_R = 1 - p_L - p_M$. In order for m to be a best response by player 2 it must be that the expected payoff of player 2 choosing m exceeds that of l and r. In other words, both the following inequalities must be satisfied:

$$\begin{aligned} 7p_M + 3\left(1 - p_M\right) &\geq 4p_M + 6p_L + 1 - p_M - p_L \\ 7p_M + 3\left(1 - p_M\right) &\geq 4p_M + p_L + 6\left(1 - p_M - p_L\right). \end{aligned}$$

At this stage, we do not know anything else about player 1, so it is fine to leave these inequalities as they are. Anyway, a little bit of simplification transforms them into

$$\begin{aligned} 3 &\geq 6p_L + p_R \\ 3 &\geq p_L + 6p_R, \end{aligned}$$

which means the probabilities of choosing L and R must be low enough. In other words, player 2 **must be thinking that M is very likely if she does, in fact, choose m**.

(c) Now let p_m be player 1's belief that player 2 chooses m, and p_l player 1's belief that player 2 chooses l. Clearly, $p_r = 1 - p_l - p_m$. If player 1 chooses M, his expected payoff is $3p_m + 6(1 - p_m)$. If he chooses L instead, he gets $7p_m + 4p_l + 9(1 - p_l - p_m)$; finally, choosing R gives him $7p_m + 4(1 - p_l - p_m) + 9p_l$. For M to be a best response given beliefs (p_l, p_m, p_r), it must be that M does better than both L and R. Hence we need both

$$\begin{aligned} 3p_m + 6(1 - p_m) &\geq 7p_m + 4p_l + 9(1 - p_l - p_m), \text{ and} \\ 3p_m + 6(1 - p_m) &\geq 7p_m + 4(1 - p_l - p_m) + 9p_l. \end{aligned}$$

Working on these inequalities is more worth it. Solve for p_l from the first inequality, and obtain

$$p_l \geq \frac{3 + p_m}{5}.$$

Solve for p_l from the second one,

$$p_l \leq \frac{2 - 6p_m}{5}.$$

Because both conditions cannot hold at the same time, M is never a best response for player 1. **In other words, there is no mental model of player 2's behavior that can justify player 1 choosing M.**

(d) From part (b), we know Player 2 could justify choosing m. For example, she could say "I thought player 1 was going to choose M with very high probability." However, *if player 2 understands player 1 is rational*, player 2 will go through the same calculation as part (c), and realize player 1 will never choose to play M. In other words, while there are mental models player 2 could build to explain a choice of m, none of them is consistent with rational behavior by player 1.

(e) In the following table, the best responses are boldfaced. Because they do not match, there is no pure-strategy Nash equilibrium.

		2		
		l	m	r
1	L	4,**6**	**7**,3	**9**,1
	M	6,4	3,**7**	6,4
	R	**9**,1	**7**,3	4,**6**

3. Splitting the Dollar.

(a) Nash Equilibrium of this game is any combination of two numbers that sums up to 10. Any combination that in sum exceeds 10 destroys value for both players. Any combination that sums up to a number less than 10 induces each player to regret not having asked for more.

(b) There is a unique Nash Equilibrium of the game, *each player chooses 5.* The logic above suggests (s_1, s_2) cannot be an equilibrium if $s_1 + s_2 < 10$. If $s_1 + s_2 \geq 10$, then the player with the smaller amount can always get more by picking a number closer to the higher amount. For example, let's imagine that players pick 7 and 8, securing payoffs of 7 and 3, respectively. In this situation, player 1 regrets not choosing 7.999. Finally, if both players choose the same number (>5), each player will regret not picking slightly less. For example, let's imagine that both players pick 7, securing payoffs of 5 each. In this situation, each player would regret not picking 6.999, which yields a higher payoff than 5. **Answers that rounded strategies to the closest cent were also fine.**

(c) If amounts must be in whole dollars, then there are four equilibria: $(5,5)$, $(6,5)$, $(5,6)$ and $(6,6)$. In all four cases, players get 5 each and cannot improve their payoffs further.

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