

Optimization Methods in Management Science

MIT 15.053, Spring 2013

PROBLEM SET 2 (FIRST GROUP OF STUDENTS)

Students with first letter of surnames **A–H**

DUE: FEBRUARY 21 , 2013

Problem Set Rules:

1. Each student should hand in an individual problem set.
2. Discussing problem sets with other students is permitted. Copying from another person or solution set is *not* permitted.
3. Late assignments will *not* be accepted. No exceptions.
4. The non-Excel solution should be handed in at the beginning of class on the day the problem set is due. The Excel solutions, if required, should be posted on the website by the beginning of class on the day the problem set is due. Questions that require an Excel submission are marked with EXCEL SUBMISSION . For EXCEL SUBMISSION questions, only the Excel spreadsheet will be graded.

Problem 1

(33 points total) Consider the following linear program:

$$\left. \begin{array}{ll} \max & x_1 + 2x_2 \\ \text{s.t.} & \\ \text{Const 1} & x_1 - x_2 \geq -2 \\ \text{Const 2} & x_1 + x_2 \leq 4 \\ \text{Const 3} & x_1 \leq 2.5 \\ \text{Const 4} & x_2 \leq 3 \\ & x_1, x_2 \geq 0. \end{array} \right\} \quad (\text{LP})$$

- (a) (5 points) Graph the feasible region of the LP. Is the feasible region unbounded?

Solution. The feasible region and objective function are sketched in Figure 1. No. The feasible region is bounded.

- (b) (4 points) Are any of the above constraints redundant? If so, indicate which one(s). (For large linear programs, eliminating redundant constraints can speed up the solution of the linear program.)

Solution. Yes, the constraint $x_2 \leq 3$ is redundant since deleting this constraint does not change the feasible region.

- (c) (4 points) Solve the LP using the graphical method. Explain your approach.

Solution. The optimal solution is clearly the point with coordinates (1, 3).

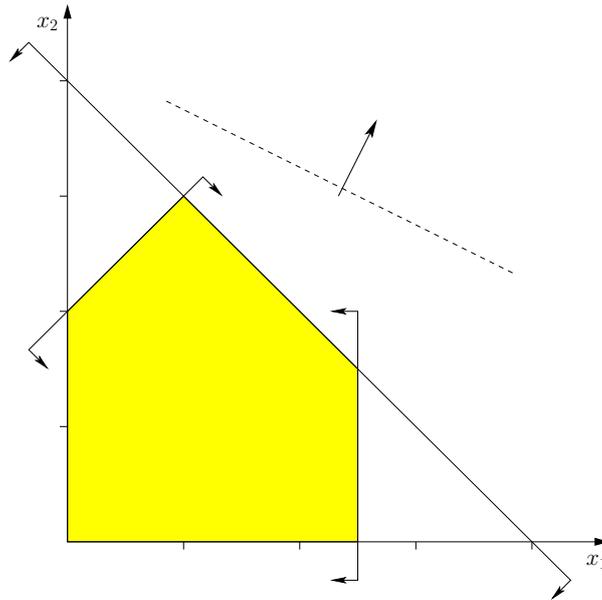


Figure 1: Feasible region for Part 1.A.

- (d) (4 points) Is there more than one optimal solution? If so give two different solutions. If not, explain using the graphical method why not?
- (e) (9 points, 3 points each) Suppose we add the constraint $2x_1 + x_2 \geq \alpha$ to (LP). For which values of α :
- is the constraint redundant?
 - the optimal solution found above is no longer optimal?
 - the problem becomes infeasible?

To answer these questions, you should use the graph of the feasible region drawn in Part (a).

Solution. The constraint $2x_1 + x_2 \geq \alpha$ has slope -2, and the feasible halfspace is the one lying to the right of the line with equation $2x_1 + x_2 = \alpha$. Therefore it is easy to see that the constraint is redundant if $2x_1 + x_2 = \alpha$ passes through the point $(0, 0)$ or if the intercept on the x_1 axis is negative. This happens for $\alpha \in [-\infty, 0]$ (for $\alpha = 0$, the line goes through the origin). A sketch of the situation is given in Figure 2. Note that the value $\alpha = 0$ is acceptable. The solution $(1, 3)$ is no longer optimal if it is cut off from the feasible region by the constraint. The line $2x_1 + x_2 = \alpha$ passes through $(1, 3)$ for $\alpha = 5$. Therefore the solution is no longer optimal for $\alpha \in (5, \infty]$ (the value $\alpha = 5$ is excluded). Finally, the problem is infeasible if all feasible points are cut off. The last point to be cut off is $(2.5, 1.5)$, for $\alpha > 6.5$. Hence the problem is infeasible for $\alpha \in (6.5, \infty]$.

- (f) (4 points) Replace the objective function $x_1 + 2x_2$ with the objective function $x_1 + \beta x_2$, and compute the values of β for which the point $(2.5, 1.5)$ is optimal.

Solution. Starting from the original objective function $x_1 + 2x_2$, which has slope $-1/2$, we easily see that we have to decrease the slope of the objective function in order to make

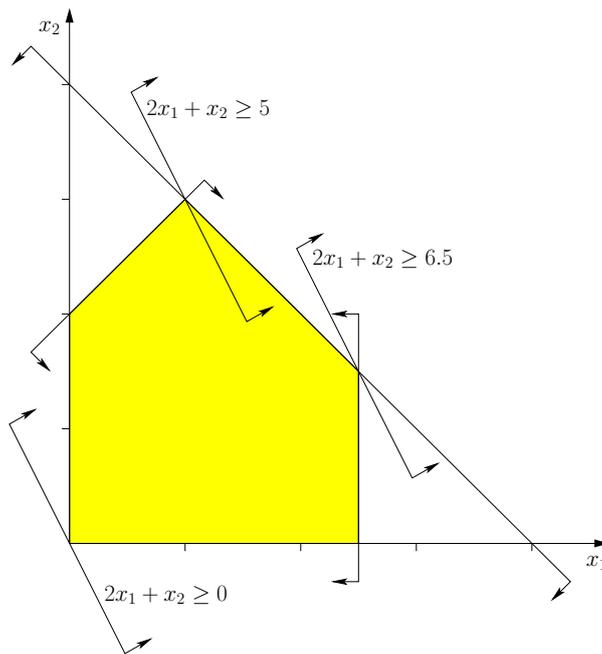


Figure 2: Initial feasible region plus constraint $2x_1 + x_2 \geq \alpha$ for Part 1.B.

$(2.5, 1.5)$ optimal. Notice that the slope is equal to $-1/\beta$. When $\beta = 1$, the objective function is parallel to the constraint $x_1 + x_2 \leq 4$ with slope -1 , and the whole segment between $(1, 3)$ and $(2.5, 1.5)$ is optimal. In order to decrease the slope from -1 , we have to decrease β . As β approaches 0, the objective function becomes a vertical line and the whole segment between $(2.5, 0)$ and $(2.5, 1.5)$ is optimal.

It follows that $(2.5, 1.5)$ is optimal for $\beta \in [0, 1]$.

Problem 2 ¹

(38 points total) A company makes three lines of tires. Its four-ply biased tires produce \$6 in profit per tire; its fiberglass belted line \$4 a tire; and its radials \$8 a tire. Each type of tire passes through three manufacturing stages as a part of the entire production process.

Each of the three process centers has the following hours of available production time per day:

	<i>Process</i>	<i>Hours</i>
1	Modeling	12
2	Curing	14
3	Assembly	16

The time required in each process to produce one hundred tires of each line is as follows:

Tire	Hours per 100 units		
	Modeling	Curing	Assembly
Four-ply	2	3	2
Fiberglass	2	2	1
Radial	4	2	2

¹This problem is based on Problem 6 of Applied Mathematical Programming, Chapter 2.

- (a) (5 points) Write a linear program to determine the optimum product mix for each day's production, assuming all tires are sold.

Solution. We define 3 decision variables that stand for the number of tires of each type to be produced each day: x_{FP} for the Fours-ply, x_{FG} for Fiberglass, and x_R for radial. The problem can be formulated as follows:

$$\left. \begin{array}{ll} \max & 6x_{FP} + 4x_{FG} + 8x_V \\ \text{s.t.} & \\ \text{Modeling :} & 0.02x_{FP} + 0.02x_{FG} + 0.04x_R \leq 12 \\ \text{Curing :} & 0.03x_{FP} + 0.02x_{FG} + 0.02x_R \leq 14 \\ \text{Assembly :} & 0.02x_{FP} + 0.01x_{FG} + 0.02x_R \leq 16 \\ & x_{FP}, x_{FG}, x_R \geq 0. \end{array} \right\}$$

- (b) (10 points) EXCEL SUBMISSION Solve the problem using the simplex algorithm, employing the Excel spreadsheet given with this problem set. Make sure you understand the formulation and the meaning of the variables, and fill in the missing coefficients of the tableau. The spreadsheet that you submit should contain the sequence of tableaus that leads to the optimal tableau *only*. You will then use the same spreadsheet to help through the rest of Problem 2, but make sure you submit a spreadsheet containing the sequence of tableaus leading to the optimal tableau for the *original* data!

- (c) (8 points) Using the Excel spreadsheet to carry out the calculations, answer the following question:

- (1) What is the initial feasible solution? Give the value for all the decision variables and all the slack variables.
- (2) What is the optimal solution? Give its objective function value and the value of all the decision variables.

Solution. The initial feasible solution is $x_{FP} = x_{FG} = x_R = 0, s_1 = 12, s_2 = 14, s_3 = 16$. The optimal solution is $x_{FP} = 400, x_{FG} = 0, x_R = 100, s_1 = 0, s_2 = 0, s_3 = 6$, with objective function value 3200.

- (d) (15 points, 5 points each) Using the Excel spreadsheet from the previous point to recompute the optimal solution of the LP, answer the following questions:

- (i) Suppose we increase the number of modeling hours per day from 12 to 13. How much does the profit increase? (Comment: This is the *shadow price* of the constraint on the modeling hours. Read the tutorial on "Sensitivity Analysis in 2 Dimensions" to learn more about shadow prices.)
- (ii) What would be the increase in the profit if we increase the number of assembly hours per day from 16 to 17 (Assume that the number of modeling hours per day is 12.)
- (iii) (Extra credit) Consider the answer to the previous two questions. What relationship do you observe between the shadow price of a constraint and the value of the corresponding slack variable in the optimal solution?

Solution.

- (i) The shadow price for the Modeling constraint is \$150 per hour of production. Therefore we could pay as much as \$150 in order to increase our production capabilities in terms of Modeling time by one hour. Any larger amount would not be profitable.
- (ii) The shadow price for the Assembly constraint is 0. It is not profitable to pay to increase the number of Assembly hours.
- (iii) Constraints such that the corresponding slack variable has zero value in the optimal solution (i.e. tight constraints) have nonzero shadow price. Constraints such that the corresponding slack variable has nonzero value in the optimal solution (i.e. non-tight constraints) have zero shadow price.

Problem 3 (First group of students)²

(34 points total) Charles Watts Electronics manufactures the following six peripheral devices used in computers specially designed for jet fighter planes: internal modems, external modems, graphics circuit boards, USB memory stick, hard disk drives, and memory expansion boards. Each of these technical products requires time, in minutes, on three types of electronic testing equipment as shown in the following table:

	Internal Modem	External Modem	Circuit Board	USB Stick	Hard Drives	Memory Boards
Test device 1	7	3	12	6	18	17
Test device 2	2	5	3	2	15	17
Test device 3	5	1	3	2	9	2

The first two test devices are available 130 hours per week. The third (device 3) requires more preventive maintenance and may be used only 100 hours each week. Watts Electronics believes that it cannot sell more than 2000, 1500, 1800, 1200, 1000, 1000 units of each device, respectively. Thus, it does not want to produce more than these units. The table that follows summarizes the revenues and material costs for each product:

Device	Revenue per unit sold (\$)	Material Cost per unit (\$)
Internal modem	200	35
External modem	120	25
Circuit board	180	40
USB memory stick	130	45
Hard disk drive	430	170
Memory expansion board	260	60

In addition, variable labor costs are \$16 per hour for test device 1, \$12 per hour for test device 2, and \$18 per hour for test device 3. Watts Electronics wants to maximize its profits. This

²This problem is based on Problem B.29 of Operations Management by Heizer and Render (2010).

problem can be formulated as follows:

$$\begin{aligned}
 \max \quad & x_I \left(200 - 35 - (7/60)16 - (2/60)12 - (5/60)18 \right) + \\
 & x_E \left(120 - 25 - (3/60)16 - (5/60)12 - (1/60)18 \right) + \\
 & x_C \left(180 - 40 - (12/60)16 - (3/60)12 - (3/60)18 \right) + \\
 & x_U \left(130 - 45 - (6/60)16 - (2/60)12 - (2/60)18 \right) + \\
 & x_H \left(430 - 170 - (18/60)16 - (15/60)12 - (9/60)18 \right) + \\
 & x_M \left(260 - 60 - (17/60)16 - (17/60)12 - (2/60)18 \right) \\
 \text{s.t.} \quad & \\
 \text{Availability Test Dev. 1 :} & \quad 7x_I + 3x_E + 12x_C + 6x_U + 18x_H + 17x_M \leq 130 * 60 \\
 \text{Availability Test Dev. 2 :} & \quad 2x_I + 5x_E + 3x_C + 2x_U + 15x_H + 17x_M \leq 130 * 60 \\
 \text{Availability Test Dev. 3 :} & \quad 5x_I + 1x_E + 3x_C + 2x_U + 9x_H + 2x_M \leq 100 * 60 \\
 \text{Max int modem demand :} & \quad x_I \leq 2000 \\
 \text{Max ext modem demand :} & \quad x_E \leq 1500 \\
 \text{Max circuit demand :} & \quad x_C \leq 1800 \\
 \text{Max USB memory demand :} & \quad x_U \leq 1200 \\
 \text{Max hard disk demand :} & \quad x_H \leq 1000 \\
 \text{Max memory demand :} & \quad x_M \leq 1000 \\
 & \quad x_I, x_E, x_C, x_U, x_H, x_M \geq 0.
 \end{aligned}$$

In this problem, there are six decision variables that indicate the number of devices as:

$$\begin{aligned}
 x_I &= \text{the number of internal modems} & x_E &= \text{the number of external modems} \\
 x_C &= \text{the number of circuit boards} & x_U &= \text{the number of USB memory sticks} \\
 x_H &= \text{the number of hard disk drives} & x_M &= \text{the number of memory expansion boards}
 \end{aligned}$$

You are given an Excel spreadsheet that solves the above problem. Suppose that the labor costs are changed to \$18 per hour for test device 1, \$15 per hour for test device 2, and \$12 per hour for test device 3. In addition, Watts Electronics considers producing external hard drives, with \$127 of revenue per unit, and \$50 of the material cost per unit. All external hard drives will be sold. The required time, in minutes, on the three testing devices are given as follows:

	<u>External hard drives</u>
Test device 1	3
Test device 2	2
Test device 3	6

All other data remains unchanged. Update the Excel spreadsheet and answer the following questions:

(a) (10 points) What is the optimal profit and the optimal production-mix?

Solution. The optimal profit is \$211420.3448 with optimal solution $x_E = 537.931$ and $x_E = 1344.828$, and all remaining variables are zero.

(b) (5 points) As you see from the solution in Part (a), no external hard drives is produced in the optimal production-mix. What is the minimum increase in the revenue of external hard

drive (in terms of \$ per unit), before they will be included in the optimal production mix? (Be accurate within one dollar.)

Solution. It will be included in an optimal solution if the revenue per unit increases to \$129.

- (c) (6 points) Suppose that Watts Electronics is considering increasing the available time of test device 2 for the next week. What would be the increase in the profit over the solution in Part (a) if the available time increases to $130 + t$ for $t = 1, 2,$ and 3 . (Assume that there are still 130 hours of test device 1 and 100 hours of test device 3.)

Solution. The increase in profit for $t = 131, 132, 133$ will be \$340.3451, \$680.069, and \$ 1020.1035, respectively.

- (d) (8 points) Based on your answer in Part (c), estimate the optimal profit if the availability time of test device 2 increases to 140. (Use Excel solver to check if your estimate is correct.) What is the formula for the optimum profit if the availability time increased by $130 + t$? (You may assume that t is between 1 and 10).

Solution. The increase in profit will be \$3400.34; the formula for profit increase in this region of t will be $\$340.03451 \times t$.

- (e) (5 points) Based on your formula in part (d), what is the contribution (the increase in optimal profit) if the availability time of test device 2 increases to 150. Use Excel solver to see if your estimation is correct (it won't). Use Excel solver to determine the maximum value of t for which your formula is correct (Be accurate to within an hour).

Solution. The formula predicts an increase of \$6800.6; however, there is only an increase of \$6192.15 in reality. The formula works $0 \leq t \leq 10$.

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