The Simplex Method for Solving a Linear Program
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Observations from Geometry

- feasible region is a convex polyhedron
- an optimum occurs at a corner point
- possible algorithm - search over corner points

Issues:

- how to identify or characterize a corner point?
- how to find a corner point to get started?
- how to search efficiently over the corner points?
- how to know when to stop?

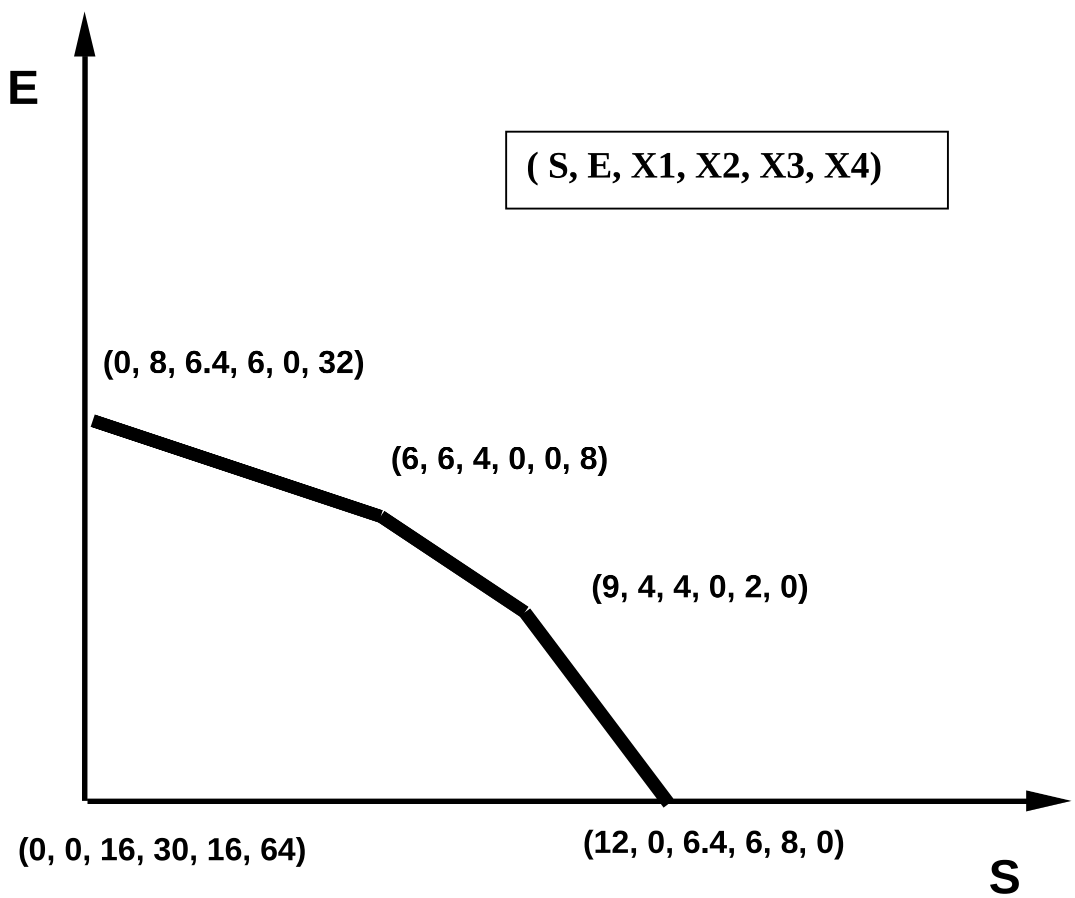
Consider Standard Form for LP:

- All constraints are expressed as equalities.
- All variables are restricted to be non negative.

ORIGINAL LP PROBLEM - ADD SLACKS AND PUT INTO STD FORM:

Max	Z =	40 S	+	50 E		
s.t.		4/5 S	+	6/5 E	<=	16
		2 S	+	3 E	<=	30
		2/3 S	+	2 E	<=	16
		16/3 S	+	4 E	<=	64

S, E >= 0



In **Standard Form**, the constraints (exclusive of the non negativity constraints) for the LP can be expressed as a linear system, **Ax = b**.

A **Basic Solution** to **Ax = b** (where **A** is an m by n matrix and m < n) is obtained by setting n - m variables equal to zero and solving the resulting m by m linear system of equations.

In a Basic Solution, the n - m variables at zero are called **non- basic variables**, while the remaining m variables are called **basic variables**.

If all of the basic variables take non negative values, then the Basic Solution is called a **Basic Feasible Solution (BFS)**.

Key Property: <u>There is a unique corner point corresponding to each Basic Feasible Solution; and there is at least one Basic Feasible Solution corresponding to each corner point (also known as extreme points).</u>

<u>Canonical Form</u>

- All constraints are expressed as equalities.
- All variables are restricted to be non negative.
- RHS's for all constraints are non negative.
- Each constraint equation has an isolated (basic) variable.

<u>Intent of the Canonical Form</u> - allows one to identify BFS's, and to move from one BFS's to another easily through a "pivot operation." To identify the BFS, we set all of the non-isolated (non basic) variables to zero, and read off the values for the isolated (basic) variables from the canonical form.

<u>Overview of Simplex Method</u>

1. Somehow find a BFS (canonical form) to start the algorithm. If you can't find a place to start, then possibly the LP formulation has no feasible solution. For most problems we actually need to solve a 'fabricated' LP to find an initial BFS for the problem of interest.

2. Determine whether or not the current BFS can be improved. This is done by asking whether there is a way to improve the objective function by making a marginal change at the corner point - namely by increasing from zero the value of one of the non isolated (non basic) variables.

3. Stop if the current BFS is optimal (no improvement possible). Otherwise, identify an adjacent BFS with a better objective value and perform the pivot operations necessary to change the canonical form. The pivot is performed by elementary row operations.

[In some cases, improvement may be possible and it may be possible to improve the objective indefinitely (i.e., make infinite profits). In such cases, the algorithm stops and reports that the solution is unbounded. You should either quit school and cash in your fortune, or take a look at the LP you have tried to solve and understand why it is mis-formulated.]

4. Repeat step 2 with the new BFS.

SIMPLEX METHOD - EXAMPLE

ORIGINAL LP PROBLEM - ADD SLACKS, PUT INTO STANDARD (AND CANONICAL) FORM

$$
\begin{array}{lllllllll}
Z - 40\,S & - & 50\,E & & & & & = & 0 \\
4/5\,S & + & 6/5\,E & + X_1 & & & & = & 16 \\
2\,S & + & 3\,E & & + X_2 & & & = & 30 \\
2/3\,S & + & 2\,E & & & + X_3 & & = & 16 \\
16/3\,S & + & 4\,E & & & & + X_4 & = & 64
\end{array}
$$

Note – from the canonical form, we identify a BFS (a corner point) by setting the non-isolated variables equal to zero. From the above, we identify the solution:

$S=0$, $E=0$, $X_1 = 16$, $X_2 = 30$, $X_3 = 16$, $X_4 = 64$.

and objective value $Z = 0$

Can we improve this solution by increasing one of the non-isolated variables?

From the objective row we see that if we increase S by 1 and hold E at 0, then Z increases by 40. Similarly, if we increase E by 1 and hold S at 0, then Z increases by 50. Thus, it appears that we can improve the solution by increasing either S or E.

Since the rate of increase is greater for E, we select E to increase.

By how much can we increase E? To answer this question we examine each of the constraint rows to see what happens to the isolated variables as we increase E, holding S at 0. To facilitate this assessment, we can re-write the constraint equations as follows (dropping S):

$X_1 = 16 - 6/5\,E$
$X_2 = 30 - 3\,E$
$X_3 = 16 - 2\,E$
$X_4 = 64 - 4\,E$

We will increase E until one of the isolated variables reaches 0 – which occurs when E = 8, and $X_3 = 0$.

This also tells us how to identify the adjacent BFS (corner point): the next corner point will have S and X_3 as non-isolated variables.

To create the next canonical form we perform a series of elementary row operations so as to replace X_3 with E in the set of isolated variables. This can be done as follows:

- Give E a coefficient of 1 by multiplying constraint 3 by ½ :

New Constraint 3: $1/3\,S + E + ½X_3 = 8$

- Eliminate E from constraint 1 by subtracting 6/5 * constraint 3 from constraint 1:

Old Constraint 1:	4/5 S	+ 6/5 E	+ X_1		=	16
-6/5* New Constraint 3:	-2/5 S	- 6/5E		-3/5 X_3	=	-48/5
New Constraint 1:	2/5 S		+ X_1	-3/5 X_3	=	32/5

- Do the same for constraints 2 and 4

- Eliminate E from objective row by adding 50* constraint 3 to the objective row:

Old Objective:	Z	- 40 S	- 50 E		=	0
50*New Constraint 3		50/3 S	+ 50 E	+ 25 X_3	=	400
New Objective:	Z	-70/3 S		+ 25 X_3	=	400

The result from these operations is a new canonical form:

$$
\begin{aligned}
Z - 70/3\,S \quad & & & + 25 X_3 & &= 400 \\
2/5\,S \quad & + X_1 & & - 3/5 X_3 & &= 32/5 \\
S \quad & & + X_2 & - 3/2 X_3 & &= 6 \\
1/3\,S \quad + E & & & + 1/2 X_3 & &= 8 \\
4\,S \quad & & & - 2 X_3 + X_4 & &= 32
\end{aligned}
$$

From the canonical form, we identify a BFS (a corner point) by setting the non-isolated variables equal to zero, namely set $S = X_3 = 0$.
Thus, we identify the solution:

$S=0$, $E=8$, $X_1 = 32/5$, $X_2 = 6$, $X_3 = 0$, $X_4 = 32$.
and objective value $Z = 400$.

Can we improve this solution by increasing one of the non-isolated variables?

From the objective row we see that if we increase S by 1 and hold X_3 at 0, then Z increases by 70/3. If we increase X_3 by 1 and hold S at 0, then Z decreases by 25. Thus, it appears that we can improve the solution, but only by increasing S.

By how much can we increase S? To answer this question we examine each of the constraint rows to see what happens to the isolated variables as we increase S, holding X_3 at 0. We re-write the constraint equations as follows (dropping X_3):

$X_1 = 32/5 - 2/5\,S$
$X_2 = 6 - S$
$E = 8 - 1/3\,S$
$X_4 = 32 - 4\,S$

We will increase S until one of the isolated variables reaches 0 – which occurs when $S = 6$, and $X_2 = 0$.

This also tells us how to identify the adjacent BFS (corner point): the next corner point will have X_2 and X_3 as non-isolated variables.

To create the next canonical form we perform a series of elementary row operations so as to replace X_2 with S in the set of isolated variables. This can be done as before:

After a series of elementary row operations to isolate S in constraint 2:

$$
\begin{array}{lllllll}
Z & & + \frac{70}{3} X_2 & - 10 X_3 & & = & 540 \\
& X_1 & - \frac{2}{5} X_2 & & & = & 4 \\
S & & + X_2 & - \frac{3}{2} X_3 & & = & 6 \\
E & & - \frac{1}{3} X_2 & + X_3 & & = & 6 \\
& & - 4 X_2 & + 4 X_3 & + X_4 & = & 8
\end{array}
$$

From this canonical form we identify the solution:

$S=6$, $E=6$, $X_1 = 4$, $X_2 = 0$, $X_3 = 0$, $X_4 = 8$.

and objective value $Z = 540$.

We see that we can improve this solution by increasing X_3, and we find that we can increase X_3 until X_4 reaches 0. We then replace X_4 by X_3 in the set of isolated variables:

$$
\begin{array}{lllllll}
Z & & + \frac{40}{3} X_2 & & + \frac{5}{2} X_4 & = & 560 \\
& X_1 & - \frac{2}{5} X_2 & & & = & 4 \\
S & & - \frac{1}{2} X_2 & & + \frac{3}{8} X_4 & = & 9 \\
E & & + \frac{2}{3} X_2 & & - \frac{1}{4} X_4 & = & 4 \\
& & - X_2 & + X_3 & + \frac{1}{4} X_4 & = & 2
\end{array}
$$

From this canonical form we identify the solution:

$S=9$, $E=4$, $X_1 = 4$, $X_2 = 0$, $X_3 = 2$, $X_4 = 0$.

and objective value $Z = 560$.

We see that we cannot improve this solution by increasing either X_2 or X_4. This is the optimal solution.