# 15.072 Queues: Theory and Application

HW 3 Solutions

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**Problem 1** Give an example of a queueing system and performance measure such that conservation law holds for admissible scheduling policies, but does not hold if the scheduler knows the service times of the jobs in the queue in advance (and thus the policy is not admissible).

HINT: work with queueing system with just one class.

**Solution:**
There are several counterexamples possible. Here is an easy one. Consider an $M/M/1$ queue with a single class. We know that the performance measure $\rho_i \mathbb{E}[S_i]$ is conserved for such a system. Since there is only one class, this means that expected waiting time $\mathbb{E}[W]$ is the same for all work-conserving and non-anticipating scheduling disciplines. However, the same is not true for anticipating service disciplines. As an example consider a work conserving scheduler in a setting where the server knows the service times of all customers precisely in advance and uses a Shortest Residual Service Time First with pre-emption(SJF) policy. Now consider two such queueing systems, one with the standard FCFS service discipline and another with SJF scheduling, and both being fed the same arrival process. Let $W_n$ and $W'_n$ denote the time customer $n$ has spent waiting in the FCFS and SJF systems respectively. It is clear that $\sum_{i=1}^{N(t)} W'_i \leq \sum_{i=1}^{N(t)} W_i \ \forall t$ when the system was idle (note that the idle periods in the two systems will be the identical). Further, the inequality is strict if there were ever 2 or more customers in the system with not all having the same service times. Since the systems are ergodic and stable, we conclude $\mathbb{E}[W_{SJF}] < \mathbb{E}[W_{FCFS}]$. Thus, the conservation law fails to hold.

**Problem 2** Suppose $X$ is a random variable distributed according to distribution function $F(x)$ and belongs to $\gamma$-MRLA class. Namely

$$\int_t^\infty \mathbb{P}(X > \tau)\, d\tau \leq \gamma \mathbb{P}(X > t) \tag{1}$$

for every $t \geq 0$.

1. Show that

$$\frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \leq \gamma.$$

HINT: integrate both parts of (1). You may assume that $\mathbb{E}\left[X^2\right] < \infty$.

2. Suppose that interarrival times $A_n$ in G/G/1 system belong to $\gamma$-MRLA class. Argue from this that the idle period $I$ in a G/G/1 queueing system satisfies

$$\frac{\mathbb{E}\left[I^2\right]}{2\mathbb{E}\left[I\right]} \leq \gamma.$$

You may assume that $I$ has finite second moment: $\mathbb{E}\left[I^2\right] < \infty$.

**Solution:**

1. Integrating both parts of (1),

$$\begin{aligned}
\int_0^x \left(\int_t^\infty \mathbb{P}\left(X > \tau\right) d\tau\right) dt &\leq \int_0^x \left(\gamma \mathbb{P}\left(X > t\right)\right) dt \\
\therefore \left[t\left(\int_t^\infty \mathbb{P}\left(X > \tau\right) d\tau\right)\right]_0^x + \int_0^x t\mathbb{P}\left(X > t\right) dt &\leq \gamma \int_0^x \mathbb{P}\left(X > t\right) dt \\
\therefore x\int_x^\infty \mathbb{P}\left(X > \tau\right) d\tau + \left[\frac{t^2}{2}\mathbb{P}\left(X > t\right)\right]_0^x + \int_0^x \frac{t^2}{2} f_X\left(t\right) dt &\leq \gamma \int_0^x \mathbb{P}\left(X > t\right) dt \\
\therefore x\int_x^\infty \mathbb{P}\left(X > \tau\right) d\tau + \frac{x^2}{2}\mathbb{P}\left(X > x\right) + \int_0^x \frac{t^2}{2} f_X\left(t\right) dt &\leq \gamma \int_0^x \mathbb{P}\left(X > t\right) dt \qquad (2)
\end{aligned}$$

Now if we take $\lim_{x\to\infty}$, we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2}{2}\mathbb{P}\left(X > x\right) &\leq \lim_{x\to\infty} \int_x^\infty t^2 f_X\left(t\right) dt \\
&= 0 \ \ldots \ \left(\because \mathbb{E}\left[X^2\right] < \infty\right) \qquad (3) \\
\Rightarrow \lim_{x\to\infty} \frac{x^2}{2}\mathbb{P}\left(X > x\right) &= 0 \\
\lim_{x\to\infty} x\int_x^\infty \mathbb{P}\left(X > \tau\right) d\tau &\leq \lim_{x\to\infty} \int_x^\infty \tau\mathbb{P}\left(X > \tau\right) d\tau \\
&= \lim_{x\to\infty} \left[\frac{\tau^2}{2}\mathbb{P}\left(X > \tau\right)\right]_x^\infty + \lim_{x\to\infty} \int_x^\infty \frac{\tau^2}{2} f_X\left(\tau\right) d\tau \\
&= 0 \ \ldots \ \left(\because \mathbb{E}\left[X^2\right] < \infty \text{ and from (4)}\right) \\
\Rightarrow \lim_{x\to\infty} x\int_x^\infty \mathbb{P}\left(X > \tau\right) d\tau &= 0
\end{aligned}$$

Then from (2), we get $\frac{\mathbb{E}\left[X^2\right]}{2\mathbb{E}\left[X\right]} \leq \gamma$, as required.

2. Consider the random variable $I_u$ indexed by parameter $u$ whose complementary distribution function is given by $\mathbb{P}(I_u > y) = \frac{\mathbb{P}(A_n > u+y)}{\mathbb{P}(A_n > u)}$. Then,

$$\begin{aligned}
\int_t^\infty \mathbb{P}(I_u > \tau)\, d\tau &= \int_t^\infty \frac{\mathbb{P}(A_n > \tau + u)\, d\tau}{\mathbb{P}(A_n > u)} \\
&= \frac{\int_{u+t}^\infty \mathbb{P}(A_n > \tau)\, d\tau}{\mathbb{P}(A_n > u)} \\
&\leq \gamma \frac{\mathbb{P}(A_n > t + u)}{\mathbb{P}(A_n > u)} \\
&= \gamma \mathbb{P}(I_u > t)
\end{aligned}$$

Thus, $I_u$ is $\gamma$ - MRLA $\forall u$. Now suppose that, at the beginning of an idle period, time $U$ elapsed since the most recent arrival $= u$. Then, the idle time $I$ has the same distribution as $I_u$. Then using the result in the previous part of this problem, and using the fact that $I_u$ is $\gamma$ - MRLA

$$\begin{aligned}
\mathbb{E}\left[I^2 | U\right] &\leq 2\gamma \mathbb{E}[I|U] \\
\Rightarrow \mathbb{E}\left[I^2\right] &\leq 2\gamma \mathbb{E}[I] \\
\Rightarrow \frac{\mathbb{E}\left[I^2\right]}{2\mathbb{E}[I]} &\leq \gamma
\end{aligned}$$

**Problem 3** Establish the following identity corresponding to the Region IV for G/M/m queueing system. Let $j < m < i+1$. Then

$$P_{ij} = \int_0^\infty \binom{m}{j} e^{-j\mu t} \left[ \int_0^t \frac{(m\mu y)^{i-m}}{(i-m)!} \left(e^{-\mu y} - e^{-\mu t}\right)^{m-j} m\mu dy \right] dA(t).$$

HINT. As in other cases condition on the duration of the interarrival time.

**Solution:**
Let $T$ be the inter-arrival time. Since the no. in the system goes down from $i+1$ to $j$, and $j < m < i+1$, the queue must become empty at some point. We condition on the time $Y$, when the queue becomes empty. Then $Y < T$. It means that in time $Y$, there were $(i+1-m)$ departures. These departures are a Poisson process with rate $m\mu$. Thus $Y$ is an Erlang $(i-m+1)$ variable and its density is given by $f_Y(y) = m\mu e^{-m\mu y} \frac{(m\mu y)^{i-m}}{(i-m)!}$. To make things complete, then we need $m-j$ departures in the remaining time $T-Y$. As $j < m$, this is a binomial distribution with success probability $p = 1 - e^{-\mu(t-y)}$. Thus,

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\begin{aligned}
P_{ij} &= \int_0^\infty \left[\int_0^t f_Y(y)\binom{m}{j}p^{m-j}(1-p)^j\,dy\right]dA(t)\\
&= \int_0^\infty \left[\int_0^t \binom{m}{j}\left(1-e^{-\mu(t-y)}\right)^{m-j}\left(e^{-\mu(t-y)}\right)^j e^{-m\mu y}\frac{m\mu\,(m\mu y)^{i-m}}{(i-m)!}dy\right]dA(t)\\
&= \int_0^\infty m\mu\binom{m}{j}e^{-j\mu t}\left[\int_0^t \left(e^{-\mu y}-e^{-\mu t}\right)^{m-j}\frac{(m\mu y)^{i-m}}{(i-m)!}dy\right]dA(t)\\
&= \int_0^\infty \binom{m}{j}e^{-j\mu t}\left[\int_0^t \frac{(m\mu y)^{i-m}}{(i-m)!}\left(e^{-\mu y}-e^{-\mu t}\right)^{m-j}m\mu dy\right]dA(t)
\end{aligned}
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