15.072 Queues: Theory and Application

HW 5 Solutions

May 12, 2006

Problem 1. Consider a Markov process with a countable state space $i = 1, 2, \ldots, n, \ldots$. Given the transition rates q_{ij} of the process derive the expected time $1/\mu_i$ that the system stays in state i and the probability p_{ij} that the next state visited after state i is state j.

Conversely, suppose you are given μ_i, p_{ij}. Obtain the values of the rates q_{ij}.

Solution:
We have,

$$
\begin{aligned}
q_{ij} &= \lim_{h\to 0} \frac{\mathbb{P}(X(t+h) = j | X(t) = i)}{h} \\
&= p_{ij} \lim_{h\to 0} \frac{\mathbb{P}(1 \text{ transition} | X(t) = i)}{h} + \lim_{h\to 0} \frac{\mathbb{P}(X(t+h) = j, > 1 \text{ transitions in } (t, t+h] | X(t) = i)}{h} \\
&= p_{ij}\mu_i \\
\therefore \sum_j q_{ij} &= \mu_i \sum_j p_{ij} \\
\Rightarrow \mu_i &= \sum_j q_{ij} \\
p_{ij} &= \frac{q_{ij}}{\sum_j q_{ij}}
\end{aligned}
$$

Note, by definition $q_{ii} = 0 \Rightarrow p_{ii} = 0$. Conversely, given p_{ij}, μ_i, $q_{ij} = p_{ij}\mu_i$, $q_{ii} \stackrel{\triangle}{=} 0$. Note that this formulae work even when $p_{ii} \neq 0$.

Problem 2. Exercise 5.3 from Chapter 5.

Solution:

(a) To show that S_1 and S_2 for tandem queues are independent we first need to understand and show the following fact about $M/M/1$ queues - if a customer enters the system at t_1 and exits the system at time t_2, then the system time $t_2 - t_1$ of that customer is independent of the departure process from the system upto time t_2. To see this consider the time reversed system which will also be an $M/M/1$ queue by reversibility. The departure process upto time t_2 of the original queue corresponds to the arrival process for the reversed queue after time (-)t_2. The system time of the customer of interest in the reversed queue is actually the same as that in the original queue. It is easy to see that (in the reversed $M/M/1$ queue), system time of a customer is independent of the arrivals that happen after its time of arrival (-)t_2 as the service discipline is FCFS. But this means that for the original system, the system time is independent of the departure process upto the time of exit from the system.

Now for the 2-tandem queueing system of the question, the departure process from the 1^{st} queue is actually the same as the arrival process for the 2^{nd} queue. Thus S_1 is independent of the arrival process to the 2^{nd} queue upto its point of arrival to the 2^{nd} queue. Since the queue length that a customer sees upon arrival will be a function of the arrival process upto its own arrival, we conclude that S_1 is independent of L_2^{+-}, which is the no. that a customer sees in Station 2 upon its arrival at Station 2. Hence S_1 must be independent of S_2. (In fact S_1 and W_2 will also be independent.)

(b) Let us denote by L_1^{--}, L_2^{--}, the no. that an arriving customer sees in stations 1 and 2 respectively, upon its arrival to the tandem system. Let L_2^{+-} be the no. that customer sees in station 2 just after completing service in station 1. Note that the event $W_1 = 0 \equiv L_1^{--} = 0$ and $W_2 = 0 \equiv L_2^{+-} = 0$. Thus

$$\begin{aligned}
\mathbb{P}(W_2 = 0) &= \mathbb{P}\left(L_2^{+-} = 0\right) \\
&= 1 - \rho_2 \\
\mathbb{P}(W_2 = 0 | W_1 = 0) &= \mathbb{P}\left(L_2^{+-} = 0 | L_1^{--} = 0\right) \\
&= \sum_{k=0}^{\infty} \mathbb{P}\left(L_2^{+-} = 0 | L_2^{--} = k, L_1^{--} = 0\right) \mathbb{P}\left(L_2^{--} = k | L_1^{--} = 0\right) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(k \text{ customers finished service at Station 2 in } S_1)\, \mathbb{P}\left(L_2^{--} = k\right) \\
&= \sum_{k=0}^{\infty} \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^k (1 - \rho_2)\, \rho_2^k \\
&= \frac{1 - \rho_2}{1 - \frac{\rho_2 \mu_2}{\mu_1 + \mu_2}} \\
&= 1 - \frac{\rho_2 \mu_1}{\mu_1 + \mu_2 - \lambda} \quad \ldots (\rho_2 = \frac{\lambda}{\mu_2}) \\
&> 1 - \rho_2 \quad \ldots (\mu_2 > \lambda) \qquad (1)
\end{aligned}$$

Thus $\mathbb{P}(W_2 = 0) \neq \mathbb{P}(W_1 = 0)$ in general. And hence W_1 and W_2 cannot be independent.

(c) Let us find $\mathbb{P}\left(L_2^{+-} = n | L_1^{--} = 0\right)$. The case $n = 0$ was solved in 2. So let us assume $n > 0$.

$$
\begin{aligned}
\mathbb{P}\left(L_2^{+-} = n | L_1^{--} = 0\right) &= \sum_{k=0}^{\infty} \mathbb{P}\left(L_2^{+-} = n | L_2^{--} = k, L_1^{--} = 0\right) \mathbb{P}\left(L_2^{--} = k | L_1^{--} = 0\right) \\
&= \sum_{k=n}^{\infty} \mathbb{P}(\text{ exactly } k-n \text{ customers finished service at Station 2 in } S_1) \, \mathbb{P}\left(L_2^{--} = k\right) \\
&= \sum_{k=n}^{\infty} \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^{n-k} \frac{\mu_1}{\mu_1 + \mu_2} (1 - \rho_2) \rho_2^k \\
&= \frac{\mu_1}{\mu_1 + \mu_2} (1 - \rho_2) \rho_2^n \sum_{k=0}^{\infty} \left(\frac{\rho_2 \mu_2}{\mu_1 + \mu_2}\right)^k \\
&= \frac{\mu_1 (1 - \rho_2)}{\mu_1 + \mu_2 - \lambda} \rho_2^n \\
\therefore \mathbb{E}[e^{-sW_2} | W_1 = 0] &= \mathbb{E}[e^{-sW_2} | L_1^{--} = 0] \\
&= \sum_{n=0}^{\infty} \mathbb{E}[e^{-sW_2} | L_2^{+-} = n, L_1^{--} = 0] \mathbb{P}\left(L_2^{+-} = n | L_1^{--} = 0\right) \\
&= \sum_{n=0}^{\infty} \mathbb{E}[e^{-sW_2} | L_2^{+-} = n] \mathbb{P}\left(L_2^{+-} = n | L_1^{--} = 0\right) \\
&= 1 - \frac{\rho_2 \mu_1}{\mu_1 + \mu_2 - \lambda} + \sum_{n=1}^{\infty} \left(\frac{\mu_2}{\mu_2 + s}\right)^n \frac{\mu_1 (1 - \rho_2)}{\mu_1 + \mu_2 - \lambda} \rho_2^n \\
&= 1 - \frac{\rho_2 \mu_1}{\mu_1 + \mu_2 - \lambda} + \frac{\mu_1 (1 - \rho_2)}{\mu_1 + \mu_2 - \lambda} \frac{\lambda}{\mu_2 + s - \lambda} \\
&= 1 - \frac{\rho_2 \mu_1}{\mu_1 + \mu_2 - \lambda} + \frac{\rho_2 \mu_1}{\mu_1 + \mu_2 - \lambda} \frac{\mu_2 - \lambda}{\mu_2 - \lambda + s} \\
\therefore \mathbb{P}(W_2 > x) &= \frac{\rho_2 \mu_1}{\mu_1 + \mu_2 - \lambda} e^{-(\mu_2 - \lambda) x}
\end{aligned}
$$

Problem 3. Exercise 5.5 from Chapter 5.

Solution:

(a) The system can be modeled as the Closed Single Class Queueing Network with N customers and 3 stations as shown in Figure 1 below

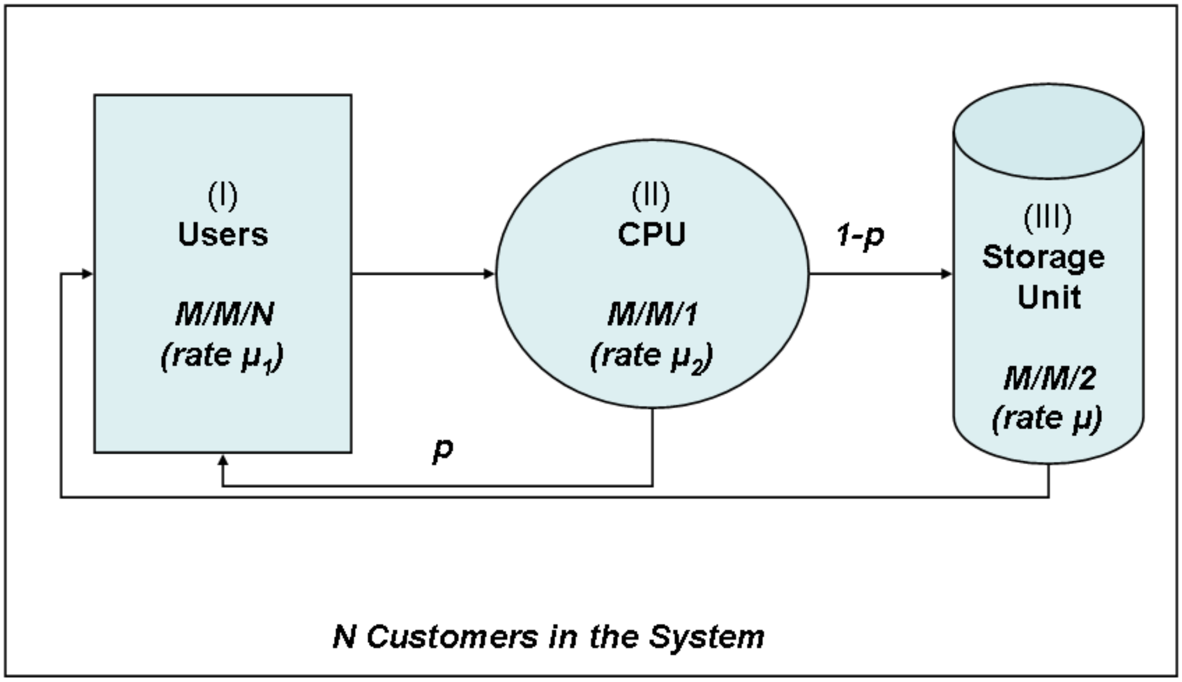

Figure 1: Closed Single Class Queueing Network for the Computer System

(b) Let us find the pseudo-arrival rates at each station. We have

$$\begin{aligned} v_2 &= v_1 \\ v_3 &= (1-p)\, v_2 \\ v_1 + v_2 + v_3 &= 1 \\ \Rightarrow v_1 = v_2 &= \frac{1}{3-p} \\ v_3 &= \frac{1-p}{3-p} \end{aligned}$$

Then,

$$
\begin{aligned}
g_i(n) &= \frac{v_i^n}{\Pi_{i=1}^n \mu_i} \\
\Rightarrow g_1(n_1) &= \frac{1}{(3-p)_1^n} \frac{1}{n_1! {\mu_1}^{n_1}} \\
g_2(n_2) &= \frac{1}{(3-p)^{n_2}} \frac{1}{{\mu_2}^{n_2}} \\
g_3(n_3) &= \left(\frac{1-p}{3-p}\right)^{n_3} \frac{1}{2^{n_3-1} \mu^{n_3}} - 1[n_3 = 0]
\end{aligned}
$$

Let $G \triangleq \sum_{n_1,n_2,n_3:n_1+n_2+n_3=N} g_1(n_1)\, g_2(n_2)\, g_3(n_3)$. Then

$$
\begin{aligned}
\mathbb{P}(L_1 = n_1) &= \frac{g_1(n_1)}{G} \left(\sum_{n_2=0}^{N-n_1} g_2(n_2)\, g_3(N - n_1 - n_2) \right) \\
\mathbb{P}(L_2 = n_2) &= \frac{g_2(n_2)}{G} \left(\sum_{n_3=0}^{N-n_2} g_3(n_3)\, g_1(N - n_2 - n_3) \right) \\
\mathbb{P}(L_3 = n_3) &= \frac{g_3(n_3)}{G} \left(\sum_{n_1=0}^{N-n_3} g_1(n_1)\, g_2(N - n_3 - n_1) \right)
\end{aligned}
$$

(c) Using the distributions derived above, one can find the expected station occupancy $\mathbb{E}[L_1], \mathbb{E}[L_2], \mathbb{E}[L_3]$. Also let $G_{N-1} \triangleq \sum_{n_1,n_2,n_3:n_1+n_2+n_3=N-1} g_1(n_1)\, g_2(n_2)\, g_3(n_3)$. Then using Little's law

$$
\begin{aligned}
\mathbb{E}[S_2] &= \frac{\mathbb{E}[L_2]}{\lambda_2} \\
&= \frac{\mathbb{E}[L_2] G}{v_2 G_{N-1}} \\
&= \frac{(3-p)\, \mathbb{E}[L_2] G}{G_{N-1}} \\
\text{similarly, } \mathbb{E}[S_3] &= \frac{(3-p)\, \mathbb{E}[L_3] G}{(1-p)\, G_{N-1}}
\end{aligned}
$$

Then expected total time to completion for a job - T is given by

$$
\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[S_2] + (1-p)\,\mathbb{E}[S_3] \\
&= \frac{(3-p)\,\mathbb{E}[L_2]G}{G_{N-1}} + (1-p)\,\frac{(3-p)\,\mathbb{E}[L_2]G}{(1-p)\,G_{N-1}} \\
&= (3-p)\,\frac{(\mathbb{E}[L_2] + \mathbb{E}[L_3])\,G}{G_{N-1}} \\
&= (3-p)\,(N - \mathbb{E}[L_1])\,\frac{G}{G_{N-1}} \\
&= \frac{N - \mathbb{E}[L_1]}{\lambda_1}
\end{aligned}
$$

The above expression has a simple interpretation. It is simply Little's Law applied to the composite system made up of stations 2 and 3 (i.e., the CPU and the Memory Unit).

Problem 4. Exercise 5.7 from Chapter 5.

Solution:

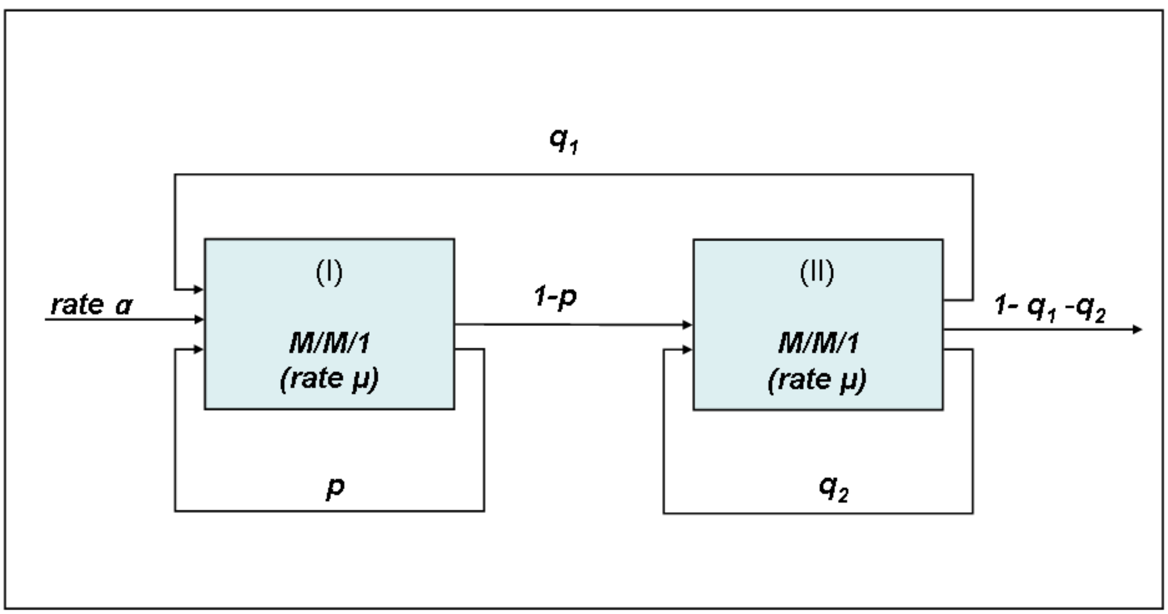

Figure 2: Problem 4 Open Single Class Network

(a)

$$\begin{aligned}
\lambda_2 &= (1-p)\,\lambda_1 + q_2\lambda_2 \\
\lambda_1 &= \alpha + p\lambda_1 + q_1\lambda_2 \\
\Rightarrow \lambda_1 &= \frac{1-q_2}{(1-q_1-q_2)\,(1-p)}\alpha \\
\lambda_2 &= \frac{1}{1-q_1-q_2}\alpha
\end{aligned}$$

An equilibrium distribution will exist if the system is stable. This requires $\mu_1 > \lambda_1$ and $\mu_2 > \lambda_2$ i.e., $\mu > \max\left(1, \frac{1-q_2}{1-p}\right) \frac{1}{1-q_1-q_2}\alpha$.

(b) We know, as Open SQNets are product form

$$\begin{aligned}
\mathbb{P} X_1 = x_1, X_2 = x_2 &= \mathbb{P}\,(X_1 = x_1)\,\mathbb{P}\,(X_2 = x_2) \\
&= (1-\rho_1)\,\rho_1^{x_1}\,(1-\rho_2)\,\rho_2^{x_2} \\
&= \left(1 - \frac{\lambda_1}{\mu}\right)\left(1 - \frac{\lambda_2}{\mu}\right)\frac{\lambda_1{}^{x_1}\lambda_2{}^{x_2}}{\mu^{x_1+x_2}}
\end{aligned}$$

(c) Let T denote the total time that an exogenous customer spends in the system. Applying Little's law to the composite system, then

$$\begin{aligned}
\mathbb{E}[T] &= \frac{\mathbb{E}[L = L_1 + L_2]}{\alpha} = \frac{\mathbb{E}[L_1] + \mathbb{E}[L_2]}{\alpha} \\
&= \frac{\lambda_1}{\alpha(\mu - \lambda_1)} + \frac{\lambda_2}{\alpha(\mu - \lambda_2)}
\end{aligned}$$

(d) Let I denote the length of an idle period and B that of a busy period. Then $\mathbb{E}[I] = \frac{1}{\alpha}$.

$$\begin{aligned}
\mathbb{P}\,(\text{system empty}) &= \frac{\mathbb{E}[I]}{\mathbb{E}[I] + \mathbb{E}[B]} \\
\therefore \mathbb{E}[B] &= \mathbb{E}[I]\left(\frac{1}{\mathbb{P}\,(X_1 = 0, X_2 = 0)} - 1\right) \\
&= \frac{1}{\alpha}\left(\frac{1}{\left(1 - \frac{\lambda_1}{\mu}\right)\left(1 - \frac{\lambda_2}{\mu}\right)} - 1\right)
\end{aligned}$$

(e) No, the total net flow in the first station is not a Poisson process in general. This is because the arrival process and departure process from an $M/M/1$ queue are not independent and their sum is in general not a Poisson process. For example when we have $q_1 = q_2 = 0$ and $p > 0$, the arrivals at station 1 will be the sum of two non-independent arrival processes which is not a Poisson process.