15.072 Queues: Theory and Application

HW 1 Solutions

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Problem 1 (a) **Exercise 1.3** Compare an $M/M/1$ system with arrival rate $\lambda/2$ and service rate μ, with an $M/M/2$ system with arrival rate λ and two servers each having rate μ in terms of the expected number of customers in each system.

Solution:

We assume that $\lambda < 2\mu$, so that both systems are stable. For the $M/M/1$ system, we have $\rho = \frac{\lambda}{2\mu}$

$$\begin{aligned}\mathbb{E}[L_a] &= \frac{\rho}{1-\rho} \\ &= \frac{\lambda}{2\mu - \lambda}\end{aligned}$$

For the $M/M/2$ system, we have $\rho_b = \frac{\lambda}{2\mu} = \rho$.

$$\begin{aligned}\mathbb{E}\left[L_b\right] &= \frac{\rho_b}{1-\rho_b} \frac{2 \cdot 2! \cdot (1+2\rho_b) \cdot (1-\rho_b)^2 + 2^2 \rho_b^2 (2+1-2\rho_b)}{(1-\rho_b)(1+2\rho_b) \cdot 2! + 2^2 \rho_b^2} \\ &= \mathbb{E}\left[L_a\right] \frac{4(1+2\rho)\left(1-2\rho+\rho^2\right) + 4\rho^2(3-2\rho)}{2(1-\rho)(1+2\rho) + 4\rho^2}\end{aligned}$$

Thus,

$$\frac{\mathbb{E}\left[L_b\right]}{\mathbb{E}\left[L_a\right]} = \frac{4}{2+2\rho} = \frac{2}{1+\rho}$$

Thus,

$$\begin{aligned}\lim_{\rho \to 0} \frac{\mathbb{E}\left[L_b\right]}{\mathbb{E}\left[L_a\right]} &= 2 \\ \lim_{\rho \to 1} \frac{\mathbb{E}\left[L_b\right]}{\mathbb{E}\left[L_a\right]} &= 1\end{aligned}$$

We note that the $M/M/2$ system is worse than $M/M/1$ in terms of the expected number in the system for all ρ. This is to be expected as the transition rate of $M/M/2$ from state

$k = 1$ to $k = 0$ is slower than that of the $M/M/1$ system, while all other transition rates are identical. Thus, the steady state probability for the states $k = 1$ and $k = 0$ is less for the $M/M/2$ system leading it to spend more time on an average in the high customers in the system states leading to a higher on an average queue length. When the traffic intensity is very low (i.e., ρ close to 0), the queues seldom see more than one customer in the system, and so only one of the two servers in the $M/M/2$ will be utilized. This is in effect halves its effective ρ thus making its average queue length twice as big that of the $M/M/1$ system. On the other extreme, when ρ is close to 1, the states $k = 0$ and $k = 1$ have low steady state probabilities and the systems have an almost identical performance.

(b) **Exercise 1.4** In a semiconductor factory a machine inspects finished products. These arrive in the machine according to a Poisson process of rate λ and are processed for a time interval which is exponentially distributed with rate μ. With probability p the parts pass the test and are ready to be used, while with probability $1 - p$ they do not pass inspection and are returned to the inspection machine to be tested again.

  (1) What is the ergodicity condition?

  (2) Find the expected number of parts in the machine.

  Solution:
  Let S_i be The test process which is independent of the service process represents a probabilistic splitting of the service process with success rate p. We know that the probabilistic splitting results in an (independent) memoryless (Poisson) process with rate $p\mu$. Since the devices that fail the test stay in the queue and their contingent service distribution is the same, the given system (processing and testing together) effectively has an exponential service rate $p\mu$. (Note that in this problem, we interpret service time distribution as the distribution of time until a product leaves the system, when the system is not empty. This is the same as the rate of transition from a state $k + 1$ to k $\forall k \geq 0$; but not quite the same as the service time distribution of a 'tagged' product.) Thus the system is like an $M/M/1$ queue with effective traffic intensity $\rho_e = \frac{\lambda}{p\mu}$.

  (1) The ergodicity condition, or the condition for existence of a steady state distribution is $\rho_e = \frac{\lambda}{p\mu} < 1$ i.e, $\lambda < p\mu$.

  (2) $\mathbb{E}[L] = \frac{\rho_e}{1-\rho_e} = \frac{\lambda}{p\mu-\lambda}$.

(c) **Exercise 1.5** The interdeparture time is the time between successive departures from a queueing system. Consider an $M/M/1$ queue with arrival rate λ and service rate μ.

  (1) Derive the probability distribution of the interdeparture time from an $M/M/1$ queue in steady-state.

  (2) Prove that the departure process from an $M/M/1$ system is Poisson with rate λ.

Solution:

(1) Let T_j be the state of the system just after a departure in the steady state. We first find the distribution of T_j. Let T_j^- denote the state of the system just before the departure. Let D_j denote the event of a departure in an interval of length Δt in the steady state. Using Baye's rule,

$$\begin{aligned}
\mathbb{P}(T_j = k|D_j) &= \mathbb{P}(T_j^- = k+1|D_j) \\
&= \frac{\mathbb{P}(T_j^- = k+1, D_j)}{\sum_{i=0}^{\infty} \mathbb{P}(T_j^- = i, D_j)} \\
&= \frac{\rho^{k+1}(1-\rho)\mu\Delta t}{\sum_{i=1}^{\infty} \rho^i(1-\rho)\mu\Delta t} \\
&= \rho^k(1-\rho)
\end{aligned}$$

Thus the state of the system just after a departure has the same distribution as the steady state of time average distribution (c.f. PASTA property). Now let X, Y denote independent exponential random variables with rates λ and μ respectively. Let Z_j denote the j^{th} interdeparture time. Then

$$\begin{aligned}
\mathbb{E}\left[e^{sZ_j}\right] &= \mathbb{E}\left[e^{sZ_j}|T_{j-1} = 0\right]\mathbb{P}\left(T_{j-1} = 0\right) + \mathbb{E}\left[e^{sZ_j}|T_{j-1} \geq 1\right]\mathbb{P}\left(T_{j-1} \geq 1\right) \\
&= \mathbb{E}\left[e^{s(X+Y)}\right](1-\rho) + \mathbb{E}\left[e^{sY}\right]\rho \\
&= \left(1 - \frac{\lambda}{\mu}\right)\frac{\lambda\mu}{(\lambda - s)(\mu - s)} + \frac{\mu}{\mu - s}\frac{\lambda}{\mu} \\
&= \frac{\lambda}{\lambda - s}
\end{aligned}$$

Thus Z_j is exponentially distributed with rate λ.

(2) In the previous part we showed that the interdeparture times are exponentially distributed with rate λ. To show that the departure process is Poisson with rate λ, we need to

show that Z_js are independent. For this we first show that Z_j and T_j are independent.

$$\begin{aligned}
& f\left(T_j = i, Z_j = z\right) \\
= & \sum_{l=0}^{l=\infty} f\left(T_j = i, Z_j = z | T_{j-1} = l\right) f\left(T_{j-1} = l\right) \\
= & f\left(T_j = i, Z_j = z | T_{j-1} = 0\right) f\left(T_{j-1} = 0\right) \\
& + \sum_{l=1}^{i+1} f\left(T_j = i | Z_j = z, T_{j-1} = l\right) f\left(T_j = i | Z_j = z\right) f\left(T_{j-1} = l\right) \\
= & (1-\rho) \int_0^z \lambda e^{-\lambda(z-x)} \mu e^{-\mu x} e^{-\lambda x} \frac{(\lambda x)^i}{i!} dx + \sum_{l=1}^{i+1} \mu e^{-\mu z} e^{-\lambda z} \frac{(\lambda z)^{i+1-l}}{(i+1-l)!} (1-\rho) \rho^l \\
= & (1-\rho) \lambda e^{-\lambda z} \int_0^{\mu z} e^{-x} \frac{(\rho x)^i}{i!} dx + \mu (1-\rho) \rho^{i+1} e^{-(\mu+\lambda) z} \sum_{l=0}^{i} \frac{(\lambda z)^l}{l!} \rho^{-l} \\
= & (1-\rho) \rho^i \lambda e^{-\lambda z} \left[\int_0^{\mu z} e^{-x} \frac{x^i}{i!} dx + e^{-\mu z} \sum_{l=0}^{i} \frac{(\mu z)^l}{l!} \right] \\
= & (1-\rho) \rho^i \lambda e^{-\lambda z}
\end{aligned}$$

Thus T_j and Z_j are independent. Now

$$\begin{aligned}
f_{Z_1 Z_2 \ldots Z_k}(z_1, z_2, \ldots, z_k) &= \sum_{t_1} f_{Z_2,\ldots,Z_k | Z_1, T_1}(z_2, \ldots, z_k | z_1, t_1) f_{Z_1, T_1}(z_1, t_1) \\
&= \sum_{t_1} f_{Z_2,\ldots,Z_k | Z_1, T_1}(z_2, \ldots, z_k | z_1, t_1) f_{Z_1}(z_1) f_{T_1}(t_1) \\
&= f_{Z_1}(z_1) \sum_{t_1} f_{Z_2,\ldots,Z_k | T_1}(z_2, \ldots, z_k | t_1) f_{T_1}(t_1) \\
&= f_{Z_1}(z_1) \cdot f_{Z_2,\ldots,Z_k}(z_2, \ldots, z_k) \\
&= f_{Z_1}(z_1) \cdot f_{Z_2}(z_2) \cdot \ldots \cdot f_{Z_K}(z_k)
\end{aligned}$$

Thus, Z_is are i.i.d. exponential distributed with rate λ and hence the departure process is Poisson.
NOTE: $M/M/1$ queues satisfy the property of 'time reversibility', a thing to be covered later. This leads quite naturally to the conclusion that the output process of an $M/M/1$ queue must be Poisson with rate λ.

(d) **Exercise 1.9** Morning joggers enter a circular ring according to a Poisson process of rate $\lambda_k = \lambda/(k+1)$, $k \geq 0$, which qualitatively captures the phenomenon that a jogger is discouraged to join the ring if there are many people using it. If they enter, they stay in the ring for an exponentially distributed time interval with mean $1/\mu$. Find the distribution of the number of joggers in steady-state.

Solution:

Let p_i be the steady state probability of the ring having i joggers. Then the probability flow-balance equantions require

$$p_{i-1}\lambda_{i-1} = p_i\mu_i$$

Here $\lambda_i = \frac{\lambda}{i+1}$ and $\mu_i = i\mu$.

$$\begin{aligned}
\therefore p_i &= \frac{\lambda}{i^2\mu}p_{i-1} \\
&= \frac{1}{(i!)^2}\left(\frac{\lambda}{\mu}\right)^i p_0
\end{aligned}$$

Normalizing,

$$\begin{aligned}
p_0 &= \left(1+\sum_{i=1}^{\infty}\frac{1}{(i!)^2}\left(\frac{\lambda}{\mu}\right)^i\right)^{-1} \\
\therefore p_i &= p_0\frac{1}{(i!)^2}\left(\frac{\lambda}{\mu}\right)^i
\end{aligned}$$

(e) **Exercise 1.12** Find the transient distribution of the number of customers in an $M/M/\infty$ queue.

Solution:

The Chapman-Kolmogorov equations for the system are

$$\begin{aligned}
\frac{dP_k(t)}{dt} &= -(\lambda_k+\mu_k)P_k(t)+\lambda_{k-1}P_{k-1}(t)+\mu_{k+1}P_{k+1}(t) \\
&= -(\lambda+k\mu)P_k(t)+\lambda P_{k-1}(t)+(k+1)\mu P_{k+1}(t) \quad \ldots(k\geq 1) \\
\frac{dP_0(t)}{dt} &= -\lambda P_0(t)+\mu P_1(t) \qquad (1)
\end{aligned}$$

Now let us define $P_i = 0\ \forall i < 0$.

$$\begin{aligned}
P(t,z) &\triangleq \sum_{k=0}^{\infty} z^k P_k(t) \\
\therefore \frac{\partial P}{\partial t} &= \sum_{k=0}^{\infty} z^k \frac{dP_k}{dt} \\
\text{and } \frac{\partial P}{\partial z} &= \sum_{k=0}^{\infty} kz^{k-1}P_k = \sum_{k=0}^{\infty}(k+1)z^k P_k
\end{aligned}$$

Multiplying the k^{th} equation in (1) with z^k and adding them we get

$$\begin{aligned}
\sum_{k=0}^{\infty} z^k \frac{dP_k}{dt} &= -\lambda \sum_{k=0}^{\infty} z^k P_k - \mu \sum_{k=0}^{\infty} k z^k P_k + \lambda \sum_{k=0}^{\infty} z^k P_{k-1} + \mu \sum_{k=0}^{\infty} k + 1 z^k P_{k+1} \\
\Rightarrow \frac{\partial P}{\partial t} &= -\lambda P - \mu z \frac{\partial P}{\partial z} + \lambda z P + \mu \frac{\partial P}{\partial z} \\
&= \left(\mu \frac{\partial P}{\partial z} - \lambda P \right) (1 - z)
\end{aligned}$$

If we make the substitution $P = \exp(f(t,z))$ then the above equation reduces to the following PDE

$$\frac{\partial f}{\partial t} + (\mu z - \mu) \frac{\partial f}{\partial z} = \lambda (z - 1) \tag{2}$$

This is in fact a standard(!) PDE and its generic solution is of the form

$$\begin{aligned}
f(t,z) &= \frac{\lambda}{\mu} (z-1) + \phi \left(e^{-\mu t} (z-1) \right) \\
\Rightarrow P(t,z) &= e^{\rho(z-1)} \Phi \left(e^{-\mu t} (z-1) \right)
\end{aligned} \tag{3}$$

The function Φ in (3) is arbitrary and is obtained by enforcing the boundary conditions. If we assume that at $t = 0$, there were no customers in the system. Then we get $P(0,z) = 1$ $\forall z$. This requires

$$\begin{aligned}
\Phi(z-1) &= e^{-\rho(z-1)} \\
\Rightarrow \Phi(u) &= e^{-\rho u} \\
P(z,t) &= e^{\rho(z-1)} e^{-\rho\left(e^{-\mu t}(z-1)\right)} \\
&= e^{-\rho\left(1-e^{-\mu t}\right)} e^{\rho\left(1-e^{-\mu t}\right)z} \\
&= e^{-\rho\left(1-e^{-\mu t}\right)} \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \left(1 - e^{-\mu t}\right)^k z^k \\
\Rightarrow P_k(t) &= e^{-\rho\left(1-e^{-\mu t}\right)} \frac{\left(\rho\left(1-e^{-\mu t}\right)\right)^k}{k!}
\end{aligned} \tag{4}$$

As a quick check, we verify that $\lim_{t\to\infty} P_k = e^{-\rho} \frac{\rho^k}{k!}$, which is the Poisson distribution, the same as the steady state distribution for $M/M/\infty$ queue.

The expression in (4) has an interesting interpretation. We note that in fact the transient distribution is Poisson, with a time modulated intensity $= \rho(1 - e^{-\mu t})$. The following alternate method of deriving the transient distribution is much more revealing about this.

Note that a customer in an $M/M/\infty$ queue gets serviced immediately and hence its system time is the same as its service time which is exponentially distributed. Now fix a time t. Consider some $s \leq t$ and a small interval $[s, s + ds)$. The no. of arrivals in this interval is Poisson with intensity λds. Also, an arrival in this interval will not be serviced by time t and hence will be still in the system with a probability $e^{-\mu(t-s)}$. Since these two events are independent, we have in fact a probabilistic splitting of the Poisson arrivals in the interval $[s, s + ds)$ and the effective feed to the no. of customers in the system at t is Poisson with intensity $\lambda e^{-\mu(t-s)}ds$. If we consider all such disjoint intervals in $[0, t]$, we note that the corresponding Poisson random variables are independent with intensities $\lambda e^{-\mu(t-s)}ds$. The sum of these independent Poisson variables is the no. of customers in the system at time t, and hence the latter must also be Poisson with its intensity $\bar{\lambda}(t)$ given by the sum-integral

$$\begin{aligned} \bar{\lambda}(t) &= \int_0^t \lambda e^{-\mu(t-s)} \\ &= \rho\left(1 - e^{-\mu t}\right) \end{aligned}$$

Hence, the distribution of the no. in the system at time t is given by (4).

Problem 2 Show that Coxian distribution with m stages has coefficient of variation at least $1/m$. Find a distribution for which the CV becomes $1/m$.

Solution:
Note that for this problem $CV \triangleq \frac{\mathrm{Var}(X)}{\mathbb{E}^2[X]}$. An m-stage Coxian Variable may be expressed as $X = I_1Y_1 + I_2Y_2 + I_2I_3Y_3 + \ldots + I_2I_3 \cdots I_mY_m$, where $I_1, I_2, \ldots, I_m$ are independent Bernoulli random variables with probability of success p_m and $Y_1, Y_2, \ldots, Y_m$ are exponential variables with

rate λ_i and these variables are independent, and $p_1 = 1$. Thus,

$$
\begin{aligned}
\mathbb{E}\left[X\right] &= \sum_{i=1}^{m} \mathbb{E}\left[Y_i \Pi_{j=1}^{i} I_j\right] \\
&= \sum_{i=1}^{m} \mathbb{E}\left[Y_i\right] \Pi_{j=1}^{i} \mathbb{E}\left[I_j\right] \\
&= \sum_{i=1}^{m} q_i \frac{1}{\lambda_i} \\
& \quad \text{where, } q_1 = 1 \text{ and } q_i = q_{i-1} p_i, i \geq 2. \\
\mathbb{E}\left[X^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{m} Y_i \Pi_{j=1}^{i} I_j\right)^2\right] \\
&= \sum_{i=1}^{m} \mathbb{E}\left[\left(Y_i \Pi_{j=1}^{i} I_j\right)^2\right] + 2 \sum_{k=1}^{m} \sum_{l=1}^{k-1} \mathbb{E}\left[Y_k Y_l \Pi_{j=1}^{k} I_j \Pi_{j=1}^{l} I_j\right] \\
&= \sum_{i=1}^{m} \mathbb{E}\left[Y_i^2\right] \Pi_{j=1}^{i} \mathbb{E}\left[I_j\right] + 2 \sum_{k=1}^{m} \sum_{l=1}^{k-1} \mathbb{E}\left[Y_k\right] \mathbb{E}\left[Y_l\right] \Pi_{j=1}^{k} \mathbb{E}\left[I_j\right] \\
&= \sum_{i=1}^{m} q_i \frac{2}{\lambda_i^2} + 2 \sum_{k=1}^{m} \sum_{l=1}^{k-1} q_k \frac{1}{\lambda_k} \frac{1}{\lambda_l} \\
\therefore \mathbb{E}\left[X^2\right] &\geq \sum_{i=1}^{m} q_i^2 \frac{2}{\lambda_i^2} + 2 \sum_{k=1}^{m} \sum_{l=1}^{k-1} q_k q_l \frac{1}{\lambda_k} \frac{1}{\lambda_l} \qquad (5) \\
&= \mathbb{E}\left[X^2\right] + \sum_{i=1}^{m} q_i^2 \frac{1}{\lambda_i^2} \\
\therefore \mathrm{Var}\left(X^2\right) &\geq \sum_{i=1}^{m} q_i^2 \frac{1}{\lambda_i^2} \\
&\geq \frac{1}{m} \left(\sum_{i=1}^{m} q_i \frac{1}{\lambda_i}\right)^2 \\
&= \frac{1}{m} \left(\mathbb{E}\left[X\right]\right)^2 \\
\Rightarrow CV &\geq \frac{1}{m}
\end{aligned}
$$

If we have an Erlang-m variable (which can be also viewed as an m-stage Coxian variable), then we note that $q_i = 1$ and $\lambda_i = \lambda \forall i$ and equality holds in all steps from (5). Thus the bound becomes tight for an Erlang distribution. It is easy to see that this is also a necessary condition.

Problem 3 Palm-Khintchine Theorem. Special case Consider a sequence of n independent renewal processes observed at infinity. $A_j = \{\tau_1^j, \tau_1^j + \tau_2^j, \ldots, \tau_1^j + \cdots + \tau_m^j, \ldots\}$. All of them have interrenewal times τ_m^j which are i.i.d. with distribution F. We rescale all of the renewal times by a factor n (that is A_j becomes $\{n\tau_1^j, n(\tau_1^j + \tau_2^j), \ldots, n(\tau_1^j + \cdots + \tau_m^j, \ldots)\}$) and consider a superposition $\bar{A}_n = \cup_{1\leq j\leq n} A_j$. Establish that $\bar{A}_n$ has in the limit a Poisson distribution:

$$\lim_{n\to\infty} \mathbb{P}(\bar{A}_n(0,t) = k) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t),$$

where $\lambda = 1/\mathbb{E}[\tau_m^j]$.

Note I. You may want to use the following fact from the renewal theory. If a renewal process with expected interrenewal time $\bar{z}$ is observed at infinity, then the time Z till the next renewal (forward recurrence time) has density $f_Z(z) = \mathbb{P}(\tau_1 \geq z)/\bar{\tau} = (1 - F(z))/\bar{\tau}$.

Note II. Palm-Khintchine Theorem holds under more general assumptions of superimposing renewal processes with different rates, provided that they "scale" similarly. Here we simplifed the statement somewhat. Also you are not required to prove that interrenewal times of $\bar{A}_n$ are asymptotically independent, that you would need to show that the limiting process is Poisson.

Solution:

Lets find the complimentary distribution function for the forward recurrence time.

$$\begin{aligned}
\mathbb{P}(Z > z) &= \lambda \int_z^\infty (1 - F(x))dx \\
&= 1 - \lambda \int_0^z (1 - F(x))dx \\
&= 1 - \lambda z + zQ(z)
\end{aligned}$$

where, we define $Q(z) = \frac{\int_0^z F(x)dx}{z}, z \neq 0$ and $Q(0) = 0$. We assume that there are no bulk-arrivals in the process A_j. Then, $\lim_{t\to 0} F(t) = 0$, Also

$$\begin{array}{cccccc}
0 & \leq & \lim_{t\to 0} \frac{\int_0^t F(x)dx}{t} & \leq & \lim\limits_{t\to 0} F(t) = 0 \\
\Rightarrow & & \lim_{t\to 0} Q(t) & & = 0
\end{array}$$

Let $\bar{X}^j$, $\bar{Z}^j$ denote the inter-arrival and rescaled forward recurrence time in the j^{th} process. Then,

$$\begin{aligned}
\mathbb{P}(A_j(0,t) = 0) = \mathbb{P}\left(\bar{Z}^j > t\right) &= \mathbb{P}\left(Z^j > \frac{t}{n}\right) \\
&= 1 - \lambda\frac{t}{n} + \frac{t}{n} Q\left(\frac{t}{n}\right) \qquad (6)
\end{aligned}$$

Further,

$$\begin{aligned} 0 &\leq \mathbb{P}\left(A_j\left(0,t\right)\geq 2\right) &\leq& \ \mathbb{P}\left(\bar{Z}_j\leq t,\bar{X}_j\leq t\right)=\mathbb{P}\left(\bar{Z}_j\leq t\right)\mathbb{P}\left(\bar{X}_j\leq t\right) \\ \therefore 0 &\leq \mathbb{P}\left(A_j\left(0,t\right)\geq 2\right) &\leq& \left(\lambda\frac{t}{n}-\frac{t}{n}Q\left(\frac{t}{n}\right)\right)F\left(\frac{t}{n}\right) \end{aligned}$$

Hence $\mathbb{P}\left(A_j\left(0,t\right)\geq 2\right)=\frac{t}{n}o_2\left(\frac{t}{n}\right)$; where, $o_2\left(\cdot\right)$ is some function satisfying $\lim_{t\to 0}o_2\left(\cdot\right)=0$.
Also,

$$\begin{aligned} \mathbb{P}\left(A_j\left(0,t\right)=1\right) &= 1-\mathbb{P}\left(A_j\left(0,t\right)=0\right)-\mathbb{P}\left(A_j\left(0,t\right)\geq 2\right) \\ &= \lambda\frac{t}{n}-\frac{t}{n}Q\left(\frac{t}{n}\right)-\frac{t}{n}o_2\left(\frac{t}{n}\right) \end{aligned}$$

Hence $\mathbb{P}\left(A_j\left(0,t\right)=1\right)=\frac{t}{n}\left(\lambda+o_1\left(\frac{t}{n}\right)\right)$; where, $o_1\left(\cdot\right)$ is some function satisfying $\lim_{t\to 0}o_1\left(t\right)=0$.

Now, we have

$$\begin{aligned} \mathbb{P}\left(\bar{A}_n\left(0,t\right)=k\right) &= \mathbb{P}\left(\bar{A}_n\left(0,t\right)=k,\max_{1\leq i\leq n}A_i\left(0,t\right)\leq 1\right)+\mathbb{P}\left(\bar{A}_n\left(0,t\right)=k,\max_{1\leq i\leq n}A_i\left(0,t\right)>1\right) \\ \therefore\quad 0 &\leq \mathbb{P}\left(\bar{A}_n\left(0,t\right)=k\right)-\mathbb{P}\left(\bar{A}_n\left(0,t\right)=k,\max_{1\leq i\leq n}A_i\left(0,t\right)\leq 1\right) \ \leq\ \mathbb{P}\left(\max_{1\leq i\leq n}A_i\left(0,t\right)>1\right) \end{aligned} \tag{7}$$

But,

$$\begin{aligned} \lim_{n\to\infty}\mathbb{P}\left(\max_{1\leq i\leq n}A_i\left(0,t\right)>1\right) &\leq \lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{P}\left(A_i\left(0,t\right)\geq 2\right) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{t}{n}o_2\left(\frac{t}{n}\right) \\ &= 0 \end{aligned}$$

Thus, from (7),

$$\begin{aligned} \lim_{n\to\infty}\mathbb{P}\left(\bar{A}_n\left(0,t\right)=k\right) &= \lim_{n\to\infty}\mathbb{P}\left(\bar{A}_n\left(0,t\right)=k,\max_{1\leq i\leq n}A_i\left(0,t\right)\leq 1\right) \\ &= \lim_{n\to\infty}{}^{n}\mathrm{C}_{k}\left(1-\lambda\frac{t}{n}-\frac{t}{n}Q\left(\frac{t}{n}\right)\right)^{n-k}\left(\lambda\frac{t}{n}+\frac{t}{n}o_1\left(\frac{t}{n}\right)\right)^{k} \\ &= \frac{\left(\lambda t\right)^k}{k!}\lim_{n\to\infty}\frac{n!}{n^k\cdot\left(n-k\right)!}\lim_{n\to\infty}\left(1-\lambda\frac{t}{n}-\frac{t}{n}Q\left(\frac{t}{n}\right)\right)^{n-k}\lim_{n\to\infty}\left(1+\frac{1}{\lambda}o_1\left(\frac{t}{n}\right)\right)^{k} \\ &= \frac{\left(\lambda t\right)^k}{k!}\exp\left(-\lambda t\right) \end{aligned}$$

Thus the no. of arrivals in time t, $\bar{A}_n(0,t)$ has a Poisson distribution. If the inter-arrival times were indpendent in this process, then it would follow that the merged process is in fact Poisson. Here's a qualitative argument for why the inter-arrival times are aysmptotically independent.

Suppose n is large. Let B_n^i denote the process formed by merging arrivals from processes $i, i+1, \ldots, n$. W.l.o.g., assume that the 1^{st} arrival actually occurred from process 1 after a time z. This arrival doesn't have any statistical bearing on any component process except process 1. Now suppose the process 1 were to die after producing the arrival. Then the next arrival then must come from B_n^2. Since n was very large, B_n^2, which is merged of $n-1$ processes produces its first arrival according to an almost exponential process and hence the residual time until it produces its first arrival doesn't depend on z and is still exponential with the same rate. We could repeat this for all arrivals, i.e., kill the contributing process when it produces an arrival. The resulting inter-arrival times in the merged process, so modified will thus be independent. They will also almost be identically distributed because n is large. By stretching the time axis by n we achieve this 'killing' of the contributing processes; as it becomes almost impossible for a process which has produced an arrival just now to do so anytime in the near future. Thus, asymptotically, the merged process has independent, identically distributed arrivals.