15.072 Queues: Theory and Application

HW 2 Solutions

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Problem 1 (a) **Exercise 2.2** Construct a counterexample for PASTA, for a queueing system with a Poisson arrival process for which the lack of anticipation assumption fails to hold.

Solution:

There are several counterexamples possible. Here is an easy one. Consider a stable $M/M/1$ queue and let $L(t)$ denote the no. in the system at time t. Let $Z(t) \stackrel{\triangle}{=} L(t+\Delta)$ for some small, positive Δ. We note that then LAA fails to hold i.e.,

$$\begin{aligned} \mathbb{P}\left(Z(t)=n|A(u), 0 \leq u \leq t, A(t+\Delta)-A(t)\right) &= \mathbb{P}\left(L(t+\Delta)=n|A(u), 0 \leq u \leq t, A(t+\Delta)-A(t)\right) \\ &\neq \mathbb{P}\left(Z(t)=n|A(u), 0 \leq u \leq t\right) \end{aligned}$$

Both the transient as well as the steady state versions of PASTA properties do not hold as result. As

$$\begin{aligned} \mathbb{P}\left(Z(t)=0|A(u), 0 \leq u \leq t, A(t+\Delta)-A(t)=1\right) &\leq 1-e^{-\mu\Delta} \\ \therefore \mathbb{P}\left(Z(t)=0|A(u), 0 \leq u \leq t, A(t+\Delta)-A(t)=1\right) &\neq \mathbb{P}\left(Z(t)=0|A(u), 0 \leq u \leq t\right) \\ &\geq e^{-\lambda\Delta}\mathbb{P}\left(L(t)\right)=0|A(u), 0 \leq u \leq t) \end{aligned}$$

Intuitively, the quantity of interest is the queue length in the immediate future. A random incidence sampling would yield the observation 0 with a finite probability; however a Poisson arrival will almost never observe such to be the case as it would almost always end up counting itself.

If you want a 'non-phony' example; consider an $M/M/1$ queue with the modification that, if there is just one job in service and none in the queue and if service completion happens before next arrival then we 'hold' the job in service till just before the next arrival. This scheme violates LAA for $Z(t)=L(t)$. From the perspective of an arriving customer though, it would see a $L(t)$ distribution exactly as an arrival to a normal $M/M/1$ queue. Thus $\mathbb{P}(L^-(t)=0)=\mathbb{P}(L^-_{FCFS}(t)=0)$. The steady state value of this is $1-\rho$. However, after the first arrival, the system is almost never empty. Hence the time average $\mathbb{P}(L=0)=0$.

(b) **Exercise 2.5** A random variable has an Erlang distribution with k phases (E_k), if it is distributed as the sum of k identical exponential random variables. Compute the functions $K(z,t)$ and $K_o(z,t)$ for a renewal process, in which the interarrival distribution is E_2 (Erlang distribution with two phases).

Solution:

Let us first find $K_0(z,t)$. An Erlang-2 arrival process can be thought of as derived from a Poisson arrival process in which we count only even arrivals. Let $N_P(t)$ denote the underlying Poisson arrival Process. Then

$$\begin{aligned}
\mathbb{P}(N_a(t) \geq k) &= \mathbb{P}(N_P(t) \geq 2k) \\
\Rightarrow \mathbb{P}(N_a(t) = k) &= \mathbb{P}(N_P(t) = 2k) + \mathbb{P}(N_P(t) = 2k+1) \\
\therefore \sum_{k=0}^{\infty} z^k \mathbb{P}(N_a(t) = k) &= \sum_{k=0}^{\infty} z^k \left(\mathbb{P}(N_P(t) = 2k) + \mathbb{P}(N_P(t) = 2k+1)\right) \\
\therefore K_0(z,t) &= \sum_{k=0}^{\infty} z^k e^{-\lambda t} \frac{(\lambda t)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} z^k e^{-\lambda t} \frac{(\lambda t)^{2k+1}}{(2k+1)!} \\
&= e^{-\lambda t}\left(\frac{e^{\sqrt{z}\lambda t} + e^{-\sqrt{z}\lambda t}}{2} + \frac{e^{\sqrt{z}\lambda t} - e^{-\sqrt{z}\lambda t}}{2\sqrt{z}}\right) \\
&= e^{-\lambda t}\left(\cosh\left(\sqrt{z}\lambda t\right) + \frac{1}{\sqrt{z}}\sinh\left(\sqrt{z}\lambda t\right)\right)
\end{aligned}$$

The Erlang arrival process has memory, but the state space has size just 2. We say the system is in phase 0 if the process $N_P(t)$ has had an even no. of arrivals so far (i.e., the forward recurrence time is Erlang 2) and is in phase 1, if the process $N_P(t)$ has had an odd no. of arrivals so far (i.e., the forward recurrence time is exponential). Let X denote the state of the system. By symmetry, $\mathbb{P}(X=0)=\mathbb{P}(X=1)$. Then,

$$\begin{aligned}
\mathbb{P}\left(N_a^*(t)=k\right) &= \mathbb{P}\left(N_a(t)=k|X=0\right)\mathbb{P}(X=0)+\mathbb{P}\left(N_a(t)=k|X=1\right)\mathbb{P}(X=1) \\
&= \frac{1}{2}\left(\mathbb{P}\left(N_P(t)\in\{2k,2k+1\}\right)+\mathbb{P}\left(N_P(t)\in\{2k-1,2k\}\right)\right) \\
\therefore K(z,t) &= \sum_{k=0}^{\infty} z^k \mathbb{P}\left(N_a^*(t)=k\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty} z^k\left(e^{-\lambda t}\frac{(\lambda t)^{2k}}{(2k)!}+e^{-\lambda t}\frac{(\lambda t)^{2k+1}}{(2k+1)!}\right) \\
&+ \frac{1}{2}\sum_{k=1}^{\infty} z^k\left(e^{-\lambda t}\frac{(\lambda t)^{2k}}{(2k)!}+e^{-\lambda t}\frac{(\lambda t)^{2k-1}}{(2k-1)!}\right)+\frac{1}{2}e^{-\lambda t} \\
&= \frac{1}{2}e^{-\lambda t}\left(\cosh\left(\sqrt{z}\lambda t\right)+\frac{1}{\sqrt{z}}\sinh\left(\sqrt{z}\lambda t\right)\right)+\frac{1}{2}e^{-\lambda t}\left(\sum_{k=0}^{\infty} z^k\frac{(\lambda t)^{2k}}{(2k)!}+\sum_{k=0}^{\infty} z^{k+1}\frac{(\lambda t)^{2k+1}}{(2k+1)!}\right) \\
&= \frac{1}{2}e^{-\lambda t}\left(\cosh\left(\sqrt{z}\lambda t\right)+\frac{1}{\sqrt{z}}\sinh\left(\sqrt{z}\lambda t\right)+\cosh\left(\sqrt{z}\lambda t\right)+\sqrt{z}\sinh\left(\sqrt{z}\lambda t\right)\right) \\
&= \frac{1}{2}e^{-\lambda t}\left(2\cosh\left(\sqrt{z}\lambda t\right)+\frac{1+z}{\sqrt{z}}\sinh\left(\sqrt{z}\lambda t\right)\right)
\end{aligned}$$

(c) **Exercise 3.1** Let Λ be the number of customers served in a busy period of an $M/GI/1$ queue. Compute $\mathrm{E}[z^\Lambda]$.
Solution:
This problem can be approached in a way similar to that of the derivation for Tacka's results. First note that the distribution of number of customers served during a busy period Λ, would be independent of the service discipline so long as it is work conserving. Consider an LCFS (with preemption) $M/G/1$ queue. Let $C_1, C_2, \ldots C_K$ be the customers that pre-empted customer C_0 that launched the busy period i.e., there were K arrivals during the time period X spanning C_0's service. Let Λ_i be the total no. of customers served between C_i starting and finishing service (including C_i itself). Let $\mathbb{E}\left[z^\Lambda\right] \triangleq \Phi(z)$. Then, as Λ_i s are iid given K;

and should have the same distribution as Λ;

$$
\begin{aligned}
\Lambda &= 1 + \sum_{i=1}^{K} \Lambda_i \\
\therefore \Phi(z) = \mathbb{E}\left[z^{\Lambda}\right] &= z\mathbb{E}\left[z^{\sum_{i=1}^{K}\Lambda_i}\right] \\
&= z\sum_{k=0}^{\infty} \mathbb{E}\left[\Pi_{i=1}^{k} z_i^{\Lambda} | K = k\right] \Pr\left(N\left(X\right) = k\right) \\
&= z\sum_{k=0}^{\infty} \Pi_{i=1}^{k}\mathbb{E}\left[z^{\Lambda_i}\right] \int_0^{\infty} \Pr\left(N\left(t\right) = k | X = t\right) f_X\left(t\right) dt \\
&= z\sum_{k=0}^{\infty} \left(\Phi\left(z\right)\right)^k \int_0^{\infty} e^{-\lambda t}\frac{\left(\lambda t\right)^k}{k!} f_X\left(t\right) dt \\
&= z\int_0^{\infty} \sum_{k=0}^{\infty} e^{-\lambda t}\frac{\left(\Phi\left(z\right)\lambda t\right)^k}{k!} f_X\left(t\right) dt \\
&= z\int_0^{\infty} e^{-\lambda\left(1-\Phi\left(z\right)\right)t} f_X\left(t\right) dt \\
\therefore \Phi\left(z\right) &= z\beta\left(\lambda\left(1 - \Phi\left(z\right)\right)\right) \qquad (1)
\end{aligned}
$$

The above expression gives an expression that implicitly yields $\Phi\left(z\right) = \mathbb{E}\left[z^{\Lambda}\right]$.

Problem 2 1) Give a counterexample of the distributional law for a system that violates FIFO, i.e. it allows overtaking.

2) Give a counterexample of a single server queueing system where the distributional law for a system that violates FIFO, i.e. it allows overtaking.

Solution:

(a) Consider an $M/M/\infty$ queue. The queue-length distribution, as we saw in HW 1 (**Exercise 1.12**) is Poisson with intensity $\rho = \frac{\lambda}{\mu}$. However $N\left(S\right)$ is geometric. This is because for an $M/M/\infty$ queue, the system time is same as the service time, and the no. of Poisson arrivals during an exponentially distributed interval will be geometric with $\mathbb{P}\left(N\left(S\right) = k\right) = p^k\left(1-p\right)$, where $p = \frac{\lambda}{\lambda+\mu}$. Thus $L \overset{d}{\neq} N\left(S\right)$. The FIFO requirment is violated and hence the distributional law fails to hold.

(b) Consider an $M/M/1$ queue with LCFS. Again this violates FIFO. We know $L_{LCFS} \overset{d}{=} L_{FCFS}$, as the state transition Markov chain for both systems is identical and the distribution is geometric with parameter ρ i.e., $\mathbb{E}\left[z^L\right] = \frac{1-\rho}{1-\rho z}$. However, this is different from $N(S)$. To see this note that $L_{LCFS} \stackrel{d}{=} \Lambda - 1$, where Λ is the no. served during a busy period as in **Exercise 3.1** of this homework. For $M/M/1$ queue $\beta(s) = \frac{\mu}{\mu+s}$. Substituting this in (1), we get.

$$\Phi(z) = z\frac{\mu}{\mu + \lambda(1 - \Phi(z))}$$

Since, $\mathbb{E}\left[z^{L+1}\right] = z\frac{1-\rho}{1-\rho z}$, doesn't satisfy this equation in $\Phi(z)$, it follows that $N(S)$ and L cannot have the same distribution, i.e., the distributional law fails to hold.
To see a simpler and more intuitive example, consider a 2-class single server queue. Arrivals from two classes are indpedndent Poisson with rates λ_1 and λ_2 respectively. The service time for Class 1 is exponentially distributed with rate μ. That for Class 2 is 0. Also, Class 2 arrivals have a priority and they get served by pre-empting a class 1 customer if necessary. This violates the FIFO requirement for the combined system. If we consider L to be the total number in the system, then L should be distributed the same as that in an $M/M/1$ queue with $\rho = \frac{\lambda_1}{\mu}$, since, type 2 arrivals do not spend anytime in the system. Then $\mathbb{P}(L=0) = 1 - \frac{\lambda_1}{\mu}$. However, if we consider $N(S)$, then since $S = 0$ w.p. $\frac{\lambda_2}{\lambda_1+\lambda_2}$, $\mathbb{P}(N(S)=0) \geq \frac{\lambda_2}{\lambda_1+\lambda_2}$. Take $\lambda_1 = 1, \lambda_2 = 10, \mu = 2$. Clearly, the two distributions have to be different and the distributional law will not hold.

Problem 3 Consider a queueing system with i.i.d. interarrival times where service time of a customer C_n is equal to $A_{n+1} = T_{n+1} - T_n$ - the interarrival time of the next customer. Assume there is exactly one customer at time 0 and the service time of this customer is T_1.

(a) Use Generalized Little's Law for this system to establish the following fact from renewal reward theory: the forward recurrence time X^* of a renewal process with i.i.d. interrenewal times $X_n, n \geq 1$ has density

$$f_{X^*}(t) = \frac{\mathbb{P}(X > t)}{\mathbb{E}[X]}.$$

This is part (b) of Exercise 3. You may assume that the probability distribution function of X_n has density.

(b) Prove that

$$\mathrm{E}[X^*] = \frac{\lambda \mathrm{E}[X^2]}{2}.$$

(c) Prove that the Laplace tranform of X^* satisfies

$$\mathrm{E}\left[e^{-sX^*}\right] = \lambda\frac{1-\alpha(s)}{s}.$$

Solution:

(a) This result can also be derived using renewal theory. An application of Generalized Little's Law will in fact follow a similar line. Take $t_n(\omega)$ to be the $(n-1)^{th}$ arrival epoch and $\tau_n(\omega) = T_n - T_{n-1}$, i.e., the inter-arrival time for the n^{th} arrival. Let $X^*(t,\omega)$ denote the forward recurrence time at time t. Fix some $y \geq 0$. If we define $f_n(t,\omega) = 1\{X^*(t,\omega) \leq y\}$ when $t \in [T_{n-1}, T_n)$ and 0 elsewhere. Then

$$\begin{aligned} g_n(\omega) &= \int_0^\infty f_n(t,\omega) \\ &= \int_{T_{n-1}}^{T_n} 1\{X^*(t,\omega) \leq y\} dt \\ &= \min(X_n(\omega), y) \end{aligned} \tag{2}$$

Now, note that T_n are the renewal epochs for this process and g_n are just rewards collected in between these epochs. Thus g_ns are i.i.d.. Then, if we apply the law of Large Numbers,

$$\begin{aligned} \bar{g}(\omega) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g_i(\omega) \\ &= \mathbb{E}[g_n(\omega)] = \mathbb{E}[\min(X_n, y)] \\ &= \int_0^y x f(x)\, dx + y(1 - F(y)) \end{aligned} \tag{3}$$

Now, consider

$$\begin{aligned} h(t,\omega) &= \sum_{n=1}^{\infty} f_n(t,\omega) \\ &= \sum_{n=1}^{\infty} 1\{X^*(t,\omega) \leq y\} \cdot 1\{T_{n-1} \leq t < T_n\} \\ &= 1\{X^*(t,\omega) \leq y\} \\ \therefore \bar{h}(\omega) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t h(u,\omega)\, du \\ &= \lim_{t\to\infty} \frac{1}{t} \int_0^\infty 1\{X^*(u,\omega) \leq y\} du \\ &= \mathbb{P}(X^* \leq y) \end{aligned} \tag{4}$$

where in (4), we used the ergodicity property of the process. Then applying Generalized

Little's Law, we get

$$\begin{aligned}\bar{h}(\omega) &= \lambda \bar{g}(\omega) \\ \Rightarrow \mathbb{P}(X^* \leq t) &= \frac{\int_0^t x f(x)\, dx + t(1 - F(t))}{\mathbb{E}[X]} \\ \Rightarrow f_{X^*}(t) &= \frac{1 - F(t)}{\mathbb{E}[X]} = \frac{\mathbb{P}(X > t)}{\mathbb{E}[X]}\end{aligned}$$

(b) Thus,

$$\begin{aligned}\mathbb{E}[X^*] &= \int_0^\infty \frac{1 - F(t)}{\mathbb{E}[X]} t dt \\ &= \lambda \left(\left[(1 - F(t)) \frac{t^2}{2} \right]_0^\infty + \int_0^\infty \frac{t^2}{2} f(t)\, dt \right) \\ &= \frac{\lambda \mathbb{E}\left[X^2\right]}{2}\end{aligned}$$

Note that, $\lim_{t \to \infty} (1 - F(t)) t^2 = 0$, if X has a finite second moment as $(1 - F(t)) t^2 \leq \int_t^\infty u^2 f(u)\, du$ and the integral $\int_0^\infty u^2 f(u)\, du$ converges.

(c)

$$\begin{aligned}\mathbb{E}\left[e^{-sX^*}\right] &= \int_0^\infty \frac{1 - F(t)}{\mathbb{E}[X]} e^{-st} dt \\ &= \lambda \left(- \left[(1 - F(t)) \frac{e^{-st}}{s} \right]_0^\infty - \int_0^\infty \frac{e^{-st}}{s} f(t)\, dt \right) \\ &= \lambda \left(\frac{1}{s} - \frac{\mathbb{E}\left[e^{-sX}\right]}{s} \right) \\ &= \lambda \frac{1 - \alpha(s)}{s}\end{aligned}$$