15.082J and 6.855J and ESD.78J

December 2, 2010

Multicommodity Flows 2

On the Multicommodity Flow Problem O-D version

K origin-destination pairs of nodes

$(s_1, t_1), (s_2, t_2), \ldots, (s_K, t_K)$

Network $G = (N, A)$

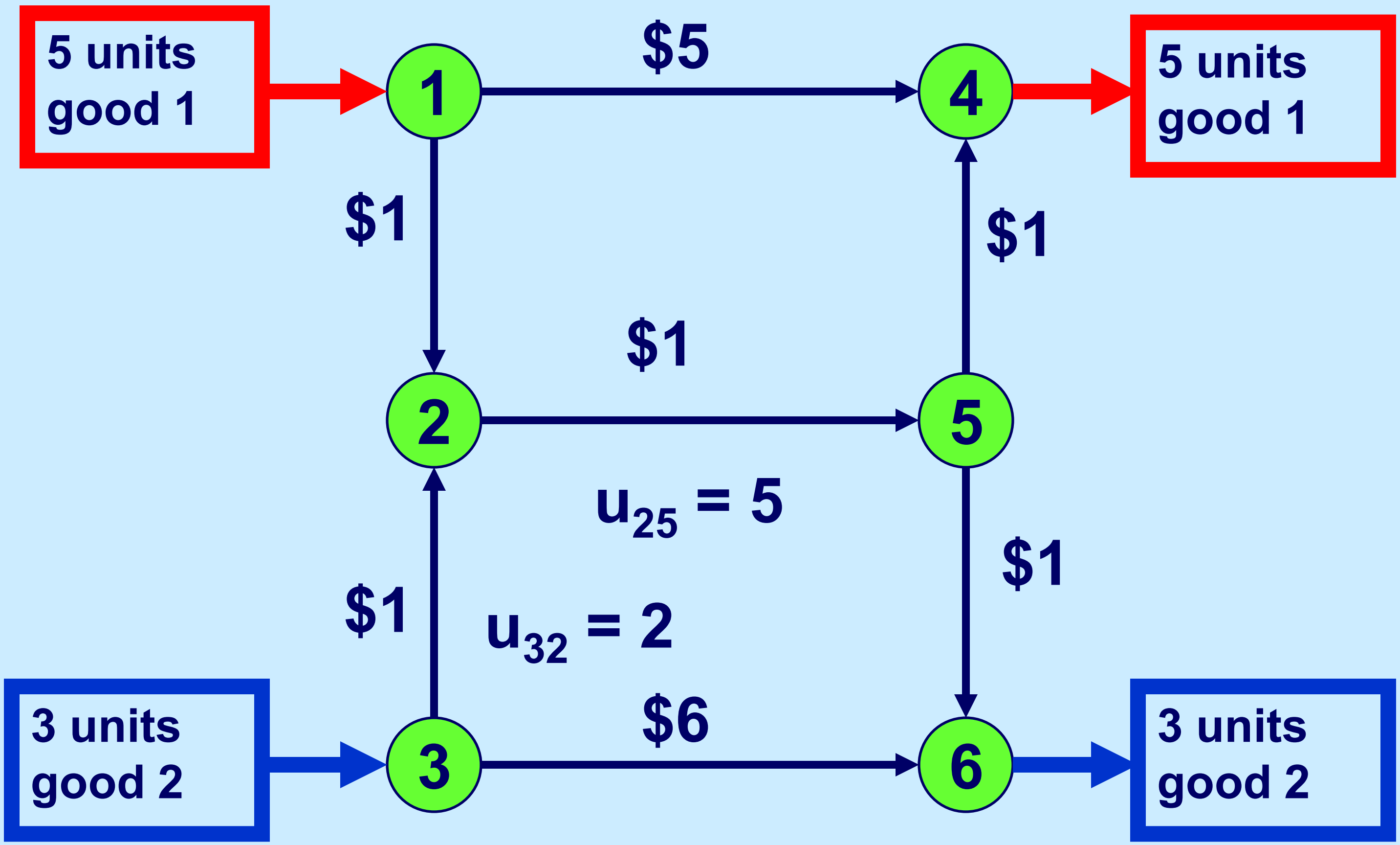
d_k = amount of flow that must be sent from s_k to t_k.

u_{ij} = capacity on (i,j) shared by all commodities

c_{ij}^k = cost of sending 1 unit of commodity k in (i,j)

x_{ij}^k = flow of commodity k in (i,j)

A Linear Multicommodity Flow Problem



The Multicommodity Flow LP

Min $$\sum_{(i,j)\in A}\sum_{k} c_{ij}^k x_{ij}^k$$

$$\sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \text{otherwise} \end{cases}$$

Supply/ demand constraints

$$\sum_k x_{ij}^k \le u_{ij} \quad \text{for all } (i,j)\in A$$

Bundle constraints

$$x_{ij}^k \ge 0 \quad \forall (i,j)\in A,\ k\in K$$

Assumptions

- **Homogeneous goods.** Each unit flow of commodity k on (i,j) uses up one unit of capacity on (i,j).
- **No congestion.** Cost is linear in the flow on (i,j) until capacity is totally used up.
- **Fractional flows.** Flows are permitted to be fractional.
- **OD pairs.** Usually a commodity has a single origin and single destination.

Optimality Conditions: Partial Dualization

Theorem. The multicommodity flow $x = (x^k)$ is an optimal multicommodity flow for (17) if there exists non-negative prices $w = (w_{ij})$ on the arcs so that the following is true

1. If $w_{ij} > 0$, then $\sum_k x_{ij}^k = u_{ij}$

2. The flow x^k is optimal for the k-th commodity if c^k is replaced by $c^{w,k}$, where

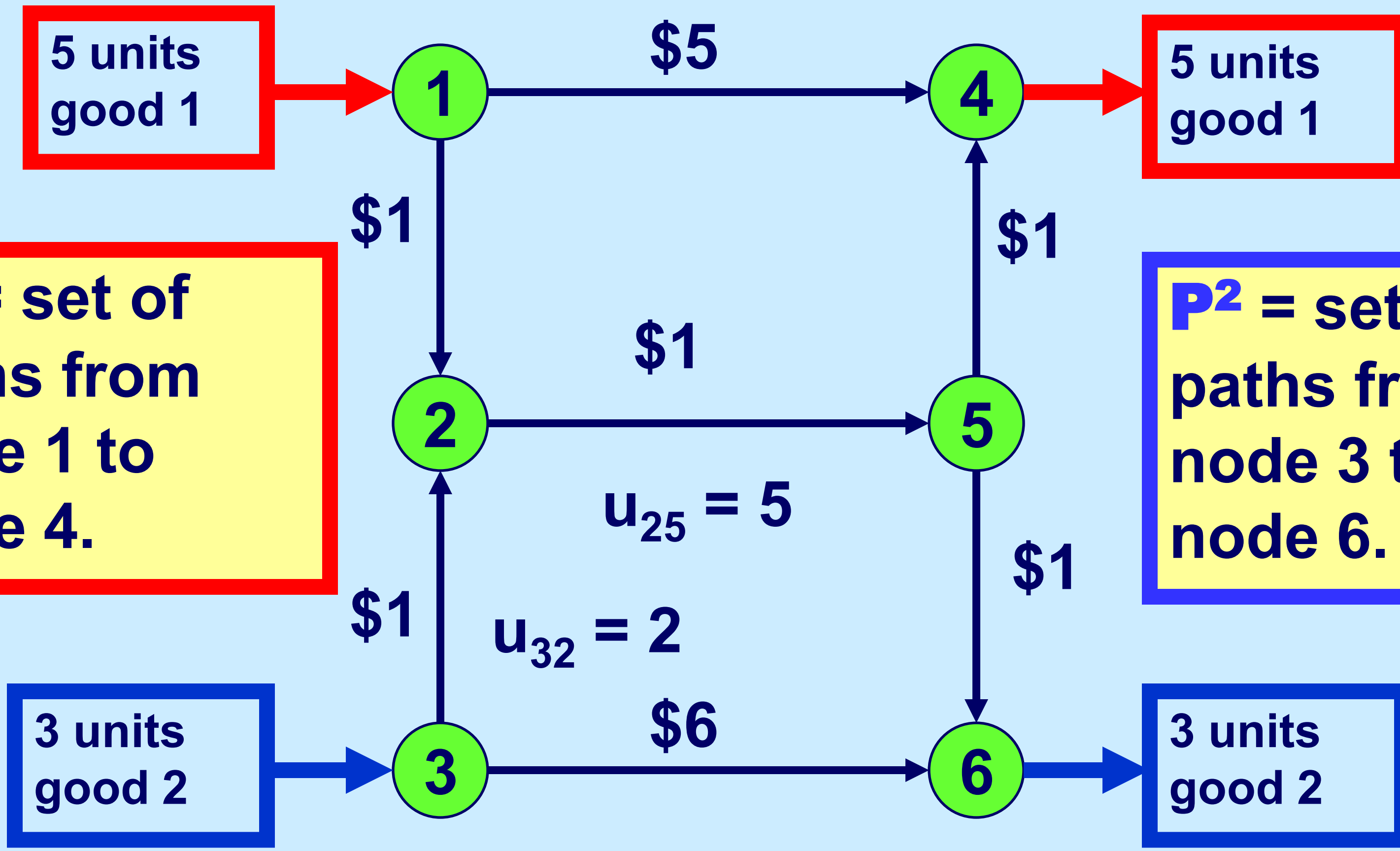
$$c_{ij}^{w,k} = c_{ij}^k + w_{ij}$$

Recall: x^k is optimal for the k-th commodity if there is no negative cost cycle in the kth residual network.

Another approach: path-based approach

- Represent flows from s_k to t_k as the sum of flows in paths.
- The resulting LP may have an exponential number of columns
- Use “column generation” to solve the LP.

A Linear Multicommodity Flow Problem



P^1 = set of paths from node 1 to node 4.

P^2 = set of paths from node 3 to node 6.

P^1 = {1-4, 1-2-5-4}

P^2 = {3-6, 3-2-5-6}

A path based formulation

f(P) = flow in path P

c(P) = cost of path P

c(1-4)	=	5
c(1-2-5-4)	=	3
c(3-6)	=	6
c(3-2-5-6)	=	3

Minimize 5 f(1-4) + 3 f(1-2-5-4) + 6 f(3-6) + 3 f(3-2-5-6)

subject to

f(1-4) + f(1-2-5-4) = 5

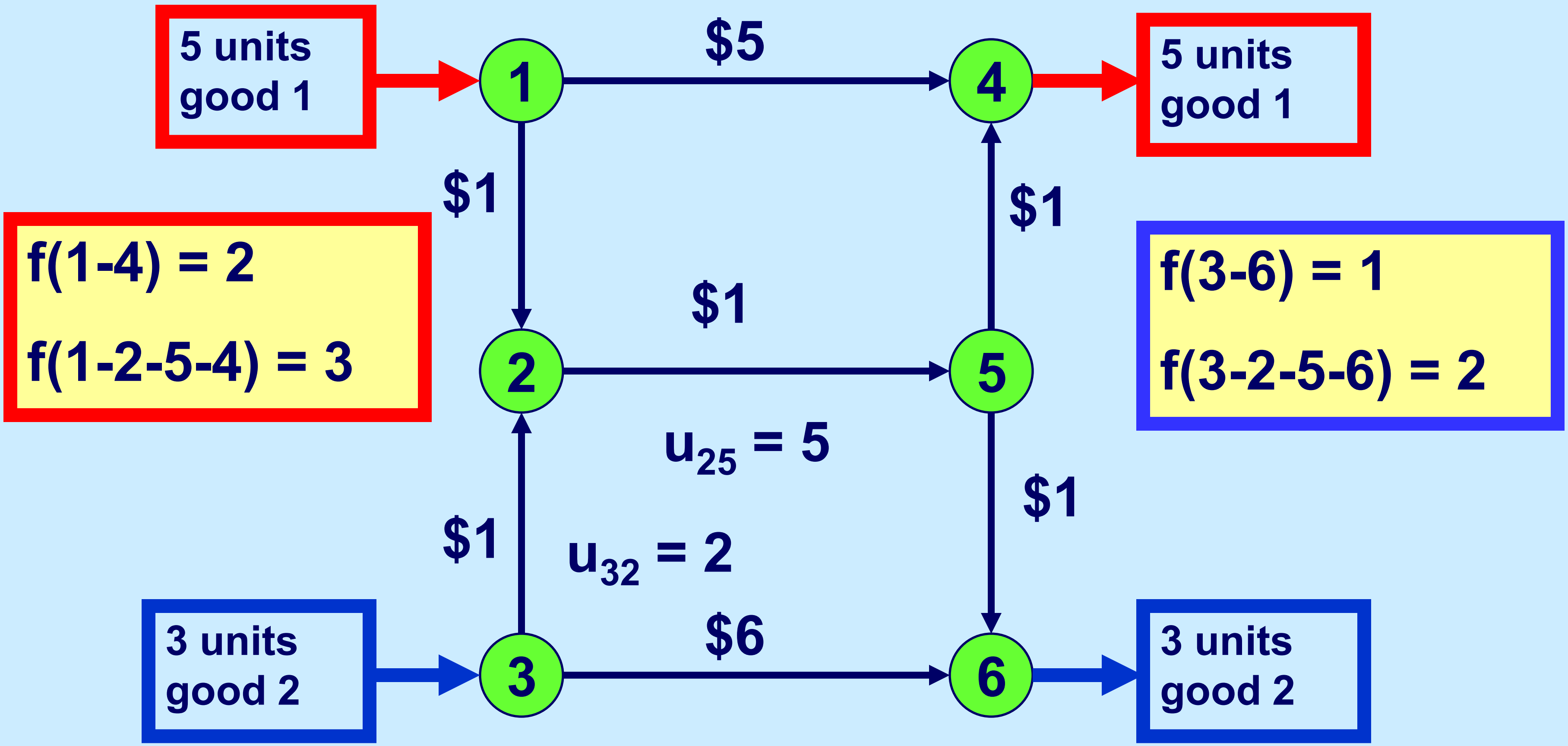
f(3-6) + f(3-2-5-6) = 3

f(1-2-5-4) + f(3-2-5-6) ≤ u_{25} = 5

f(3-2-5-6) ≤ u_{32} = 2

f(P) ≥ 0 for all paths P

Optimal solution for the path based version



The path based LP can be solved using the simplex method.

General formulation for the path based version

Let $\mathbf{P}^k$ = set of directed paths from s_k to t_k

Let $c^k(P)$ = cost of path $P \in \mathbf{P}^k$.

Let $f(P)$ = flow on path P.

$$\text{Let } \delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

Master Problem

$$\text{Minimize} \quad \sum_k \sum_{P \in \mathbf{P}^k} c^k(P) f(P)$$

$$\sum_k \sum_{P \in \mathbf{P}^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i,j) \in A$$

$$\sum_{P \in \mathbf{P}^k} f(P) = d^k \qquad \text{for } k = 1 \text{ to } K$$

$$f(P) \ge 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} \mathbf{P}^k$$

$$\text{Minimize} \quad \sum_{k} \sum_{P \in \boldsymbol{P}^k} c^k(P) f(P)$$

$$\sum_{k} \sum_{P \in \boldsymbol{P}^k} \delta_{ij}(P) f(P) \le u_{ij} \quad \text{for all } (i, j) \in A$$

$$\sum_{P \in \boldsymbol{P}^k} f(P) = d^k$$

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bundle constraints: one for each capacitated arc.

supply demand constraints: one for commodity.

variables: one for each path from origin to destination

On the path-based formulation

- m + K constraints
- exponentially many variables
- There is some optimum solution where at most m + K paths have positive flow?

Key questions

1. How can one recognize if a solution is optimal?
2. How can one deal with an LP with exponentially many variables?

FACT: One can use linear programming to optimize over the path based formulation if there are not too many paths?

Optimality Conditions: Partial Dualization

Theorem. The multicommodity flow $x = (x^k)$ is an optimal multicommodity flow for (17) if there exists non-negative prices $w = (w_{ij})$ on the arcs so that the following is true

1. If $w_{ij} > 0$, then $\sum_k x_{ij}^k = u_{ij}$

2. The flow x^k is optimal for the k-th commodity if c^k is replaced by $c^{w,k}$, where

$$c_{ij}^{w,k} = c_{ij}^k + w_{ij}$$

Recall: x^k is optimal for the k-th commodity if there is no negative cost cycle in the kth residual network.

The Restricted Master Problem

Let $\mathcal{S}^k$ = subset of $\mathcal{P}^k$ = directed paths from s_k to t_k

Let $c^k(P)$ = cost of path $P \in \mathcal{S}^k$.

Let $f(P)$ = flow on path P.

$$\text{Let } \delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

Restricted Master Problem

$$\text{Minimize} \quad \sum_k \sum_{P \in \mathcal{S}^k} c^k(P) f(P)$$

$$\sum_k \sum_{P \in \mathcal{S}^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i,j) \in A$$

$$\sum_{P \in \mathcal{S}^k} f(P) = d^k \qquad \text{for } k = 1 \text{ to } K$$

$$f(P) \ge 0 \qquad P \in \mathcal{S} = \bigcup_{k=1}^{K} \mathcal{S}^k$$

Recognizing Optimality

Let f_S be the optimal set of flows for the restricted master and let $w = W_S$ the optimum tolls (prices) on arcs.

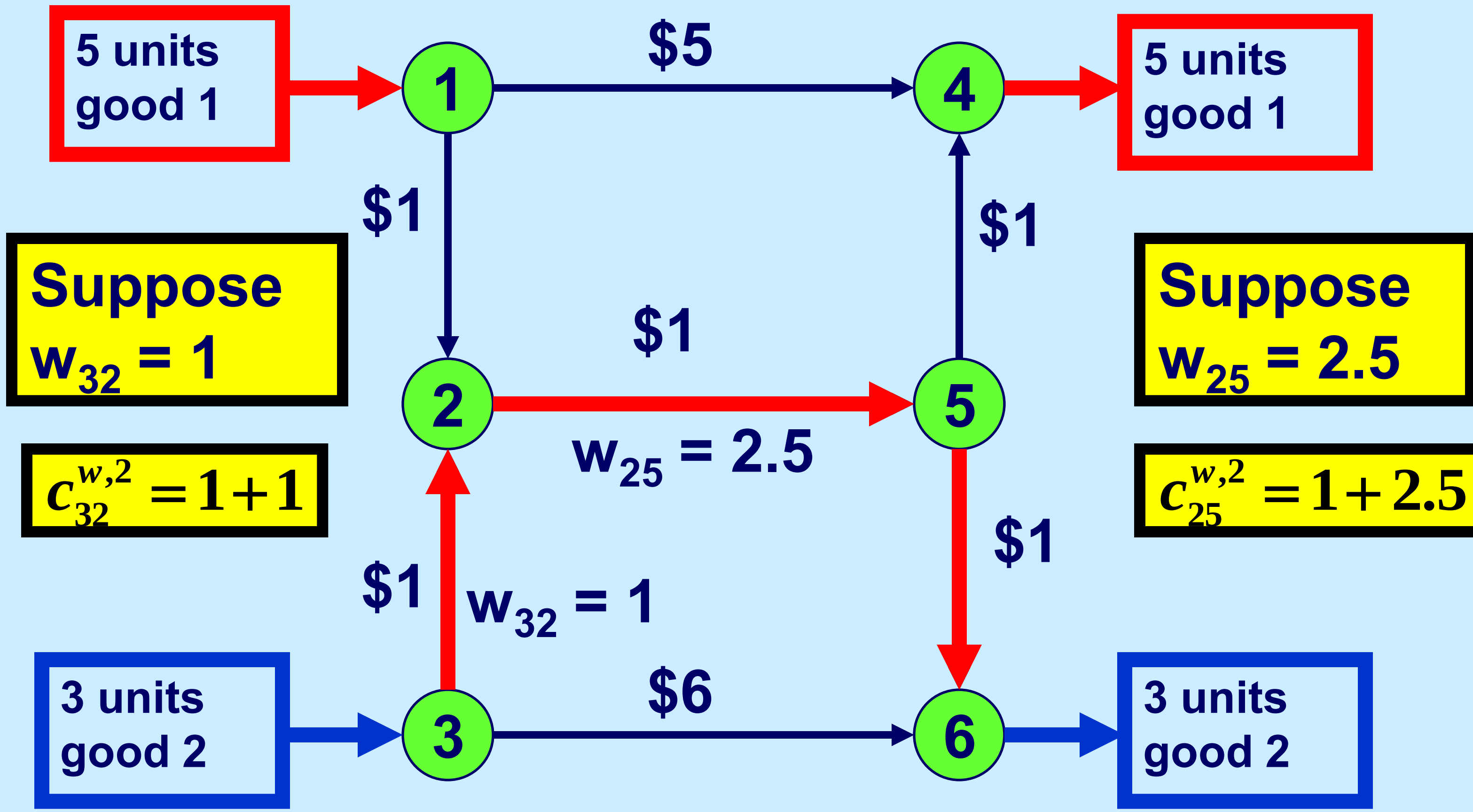
FACT: If $w_{ij} > 0$, then $\sum_P \delta_{ij} f_S(P) = u_{ij}$

Let $c_{ij}^{w,k} = c_{ij}^k + w_{ij}$ $\qquad$ $c^{w,k}(P) = \sum_{(i,j)\in P} c_{ij}^{w,k}$

Theorem: f_S is optimum for the multicommodity flow problem if it is feasible and if the following is true:

If $f_S(P) > 0$ and $P \in \mathbf{S}^k$, then P is a shortest path in from s_k to t_k with respect to $\mathbf{P}^k$.

Illustration of definitions



$$c_{32}^{w,2} = 1 + 1$$

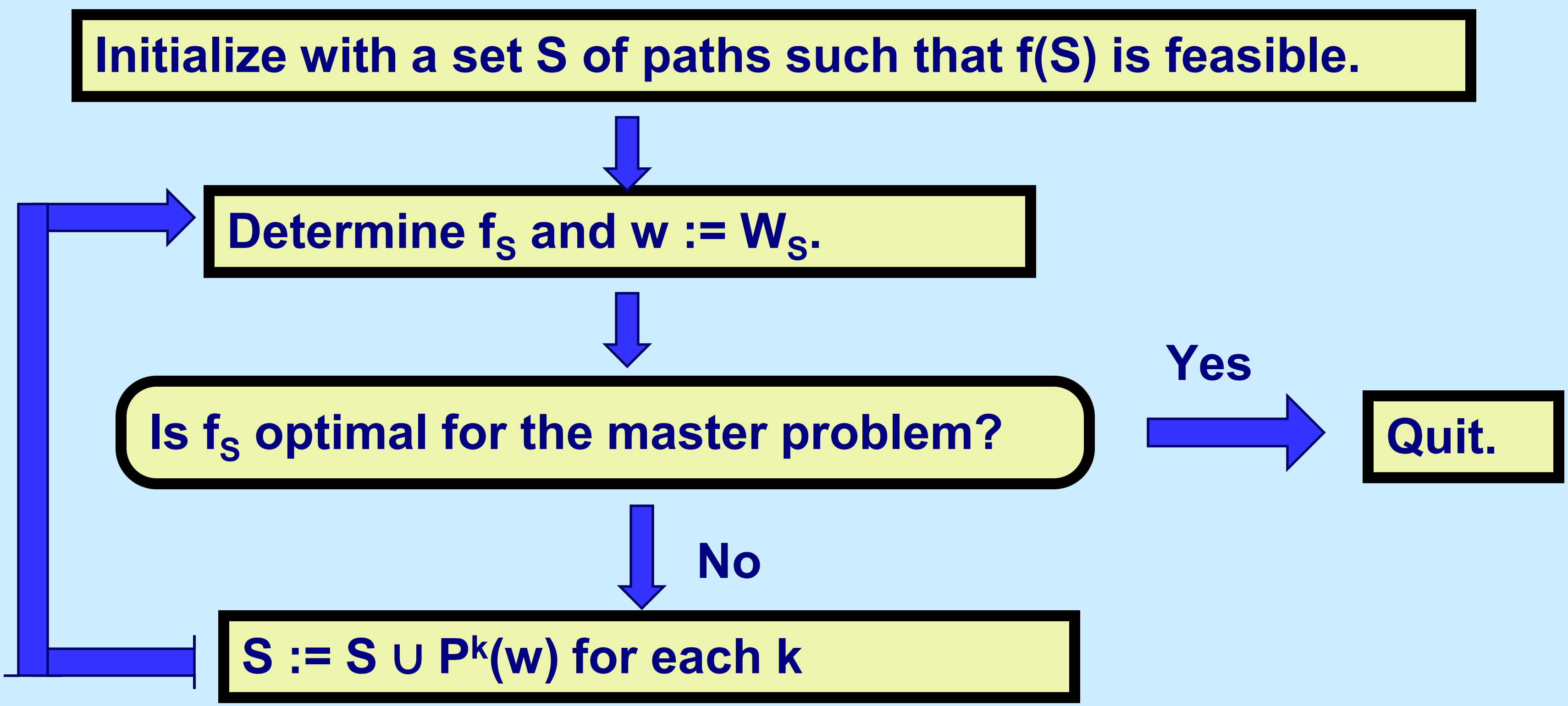
$$c_{25}^{w,2} = 1 + 2.5 = 3.5$$

$$c^{w,2}(3-2-5-6) = 2 + 3.5 + 1 = 6.5$$

Constraint Generation for Solving the Master Problem

Let f_S be the optimal set of flows and W_S the optimum tolls (prices) on arcs for the restricted master over set S.

Let $P^k(w)$ be a shortest path from s_k to t_k using costs $c^{w,k}$



Solving the Master Problem

1. Initialize $\mathcal{S}^k$ for each k.
2. Solve the restricted master problem for paths in $\mathcal{S} = \cup_k \mathcal{S}^k$ obtaining solution $x = (x^k)$.
3. Check to see if x is optimal for the master problem. If not, find new paths to add to $\mathcal{S}$ and return to step 2.

Summary of Method

- Convert multicommodity flow problem to a problem on paths

- Solve the path problem over a set S. Check if the solution is optimal for the original problem; if not add one or more paths to S, and repeat.

Comments

- One can initialize with artificial paths with infinite capacity and very high costs.
- This approach was developed by Ford and Fulkerson, and generalized by Dantzig and Wolfe to LP's
- Is often very efficient for getting close to the optimum solution. It slows down and converges slowly as more paths are generated

Mental Break

The expression “second string” means “replacement”, as in “he is a second string quarterback”. Where does this expression come from?

Archers in the middle ages carried a second string in case the first string of their bow broke.

How many different times is the Red Sea mentioned in the Bible?

0 times.

What are the four horsemen of the Apocalypse?

Conquest, slaughter, famine, and death.

Mental Break

What do the following Popes have in common: Pope Boniface IX, and Pope Benedictine XIII, and Pope Alexander V?

They all served at the same time in 1400 and were all "infallible." They were selected by vying factions of the Catholic Church during the great schism. Two of them are now referred to as "Anti-popes."

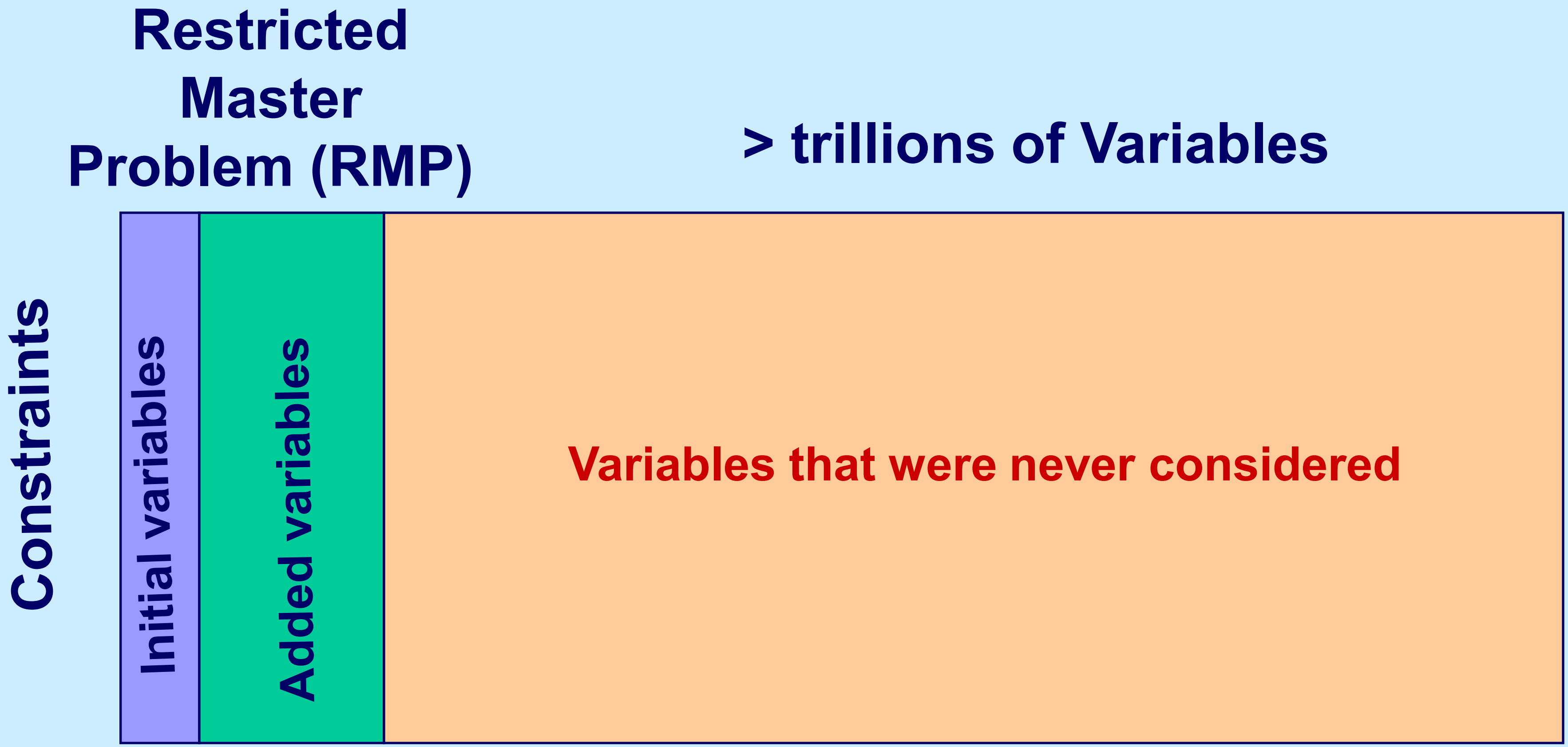
When was toilet paper first used?

In 1391 in China. They were sheets that were 2 ft. by 3 ft.

When was the first time that a magician sawed a woman in half?

1799. The magician was the Count de Grisley.

Column Generation



A story of shared resources.

Tina and Donald head separate divisions for XYZ industries. They have been asked to come up with monthly plans for their divisions.

If they knew all the resources that they had, their planning problem could be written as a linear program. But they have to share resources. The difficulty arises because they are really bad at negotiating with each other.

Their LPs

The LPs (ignoring the shared constraints) are:

$$\min \quad ax$$
$$\text{s.t.} \quad x \in X$$

Tina's LP

$$\min \quad cy$$
$$\text{s.t.} \quad y \in Y$$

Donald's LP

Shared resources:

$$Tx + Dy \le b$$

I know how to make good decisions. Let me decide what resources to use.

Donald Trump

I trust your decisions as much as I trust your choices of hair style.

Perhaps someone else could help mediate.

Tina Brown

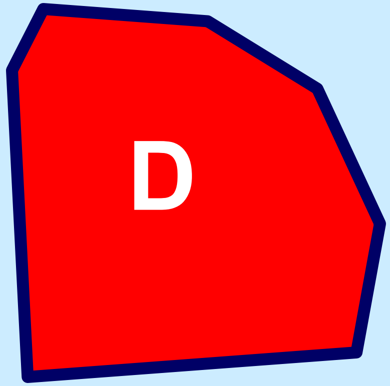
Perhaps I can help. I'll use LP Dantzig-Wolfe decomposition.

George Dantzig

Your LPs are both bounded. Any solution can be represented as a convex combination of extreme points. Just send me extreme points, and I'll find the best solution.

George Dantzig

$$\min \quad ax + cy$$
$$\text{s.t.} \quad Tx + Dy \le b$$



$$x = \sum_i \lambda_i X^i$$

$$y = \sum_j \mu_j Y^j$$

$$\sum_i \lambda_i = 1 \qquad \sum_j \mu_j = 1$$

$$\lambda \ge 0, \quad \mu \ge 0$$