

Final Exam

Instructions:

- This is a 3-hour exam.
- Please submit the exam itself and all answer booklets you have used.
- You may use the textbook, the lecture notes, your homeworks and homework solutions.

Problem 1. (2+2+2+2+2+2+2+2+2=18 points)

Indicate whether each of the following statements is true or false. No justification is required.

- (a) Integer programming problems can be solved in pseudo-polynomial time by enumeration.
- (b) If there is a pseudo-polynomial-time algorithm for some NP-complete problem, then there is a pseudo-polynomial-time algorithm for all problems in NP.
- (c) The ellipsoid method is a polynomial-time algorithm for solving integer programming problems.
- (d) The decision problem “Given a matrix $A \in \mathbb{Z}^{m \times n}$, is A totally unimodular?” is in co-NP.
- (e) If H is a rational affine half-space, $H = \{x : cx \leq \delta\}$, where c is a nonzero vector with integer components, then $H_I = \{x : cx \leq \lfloor \delta \rfloor\}$.
- (f) If $Ax \leq b$ is a totally dual integral (TDI) system, the polyhedron $\{x : Ax \leq b\}$ is integral.
- (g) If $Ax \leq b$ is a TDI system, the polyhedron $\{y : yA = c, y \geq 0\}$ is integral, for any integer right-hand side vector c , if it is not empty.
- (h) Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. For every integer point x^* in P , there exists an objective function vector c such that x^* is the unique optimal solution to $\max\{cx : x \in \text{conv}(P \cap \mathbb{Z}^n)\}$.
- (i) Let $P \subseteq \mathbb{R}^n$ be a polyhedron. For every 0/1-point $x^* \in P$, there exists an objective function vector c such that x^* is the unique optimal solution to $\max\{cx : x \in \text{conv}(P \cap \{0, 1\}^n)\}$.

Problem 2. (5 + 9 = 14 points)

Let C be the convex hull of all 0/1-solutions of the knapsack problem $\sum_{i=1}^n a_i x_i \leq b$ where $a \in \mathbb{Z}_+^n$ and $b \in \mathbb{Z}_+$.

- (a) Prove that for every $\lambda \in \mathbb{Z}_+$ the inequality

$$\sum_{i=1}^n \left\lfloor \frac{a_i}{\lambda} \right\rfloor x_i \leq \left\lfloor \frac{b}{\lambda} \right\rfloor$$

is valid for C .

- (b) Suppose that $a_1 = \dots = a_n = \lambda$, and $b/\lambda \notin \mathbb{Z}$ and $b < \lambda n$. Prove that

$$\sum_{i=1}^n x_i \leq \left\lfloor \frac{b}{\lambda} \right\rfloor$$

is a facet of C .

Problem 3. (3+3+3+3+3+3+3=21 points)

Consider the following problem. Given a directed graph $D = (V, A)$ and a weight vector $w : A \rightarrow \mathbb{R}$, find a maximum-weight subset $B \subseteq A$ such that no node of V is at the same time the head of one and the tail of another arc in B . The weight of B is $w(B) = \sum_{a \in B} w_a$.

- Formulate this problem as a 0/1-integer programming problem.
- Is the polyhedron defined by the linear programming relaxation of your 0/1-formulation integral? Why, or why not?
- Is the situation in (b) different when the underlying graph is bipartite? Why, or why not?
- For the general problem (i.e., D is an arbitrary directed graph), give an additional class of inequalities, which is not implied by the inequalities of your linear programming relaxation, but which is valid for all its 0/1-solutions. (Hint: Think about certain directed cycles in D .)
- Give an algorithm that solves the new linear programming relaxation in polynomial time.
- Is the polyhedron defined by the new linear programming relaxation integral? Why, or why not?
- Do you see any connection between the problem discussed here and the stable set problem in undirected graphs?

Problem 4. (5+5+5=15 points)

An instance of the *Steiner tree problem* is given by an undirected graph $G = (V, E)$, a cost function $c : E \rightarrow \mathbb{Z}_+$, and a partition of V into two disjoint sets R and S . The goal is to find a tree in G of minimum total cost that contains all vertices in R and an arbitrary subset of vertices in S . In the *metric Steiner tree problem*, we additionally know that costs satisfy the triangle inequality:

$$c_{uw} \leq c_{uv} + c_{vw} \text{ for all } \{u, w\}, \{u, v\}, \{v, w\} \in E.$$

- Show that any instance I of the Steiner tree problem can be solved by reduction to an instance I' of the metric Steiner tree problem. (Hint: Take I , construct an instance I' of the metric Steiner tree problem, argue that the cost of an optimal Steiner tree T' in I' does not exceed that of an optimal Steiner tree in I , and show how to obtain, starting from T' , a Steiner tree T of I of at most the same cost as T' .)
- Consider the metric Steiner tree problem on a complete graph. Prove that the cost of a minimum spanning tree on R is at most twice that of an optimal Steiner tree.
- Show that the analysis in (b) is asymptotically tight. (Hint: There exist instances with n required vertices and 1 Steiner vertex such that the cost of a minimum spanning tree is $2(n-1)$, whereas the cost of an optimal Steiner tree is n .)

Problem 5. (5+10=15 points)

In a production system there is demand $d_t \in \mathbb{R}_+$ that must be satisfied in periods $t = 1, \dots, T$, either by producing in period t or by inventory carried over from previous periods. The overall objective is to minimize the total cost, which has three components in every period t : per-unit production costs p_t , fixed costs f_t , and per-unit inventory costs h_t . The decision variables are y_t , the number of items produced in period t , x_t is one if we produce in period t , and zero otherwise, and s_t is the number of items held in inventory at the end of period t .

- (a) Give a mixed-integer optimization formulation of the problem.
- (b) Let \mathcal{F} be the set of feasible solutions. For any $k \leq T$ and $C \subseteq \{1, \dots, k\}$ show that the inequality

$$\sum_{i \in C} y_i \leq \sum_{i \in C} \left(\sum_{t=i}^k d_t \right) x_i + s_k$$

is valid for $\text{conv}(\mathcal{F})$.

Problem 6. (6+6+5=17 points)

- (a) Consider the 2-variable mixed-integer set $S = \{(x, y) \in \mathbb{Z} \times \mathbb{R}_+ : x - y \leq b\}$.

- (i) Draw the feasible region S .
- (ii) Let $f_0 = b - \lfloor b \rfloor$. Show that the inequality

$$x - \frac{1}{1 - f_0} y \leq \lfloor b \rfloor$$

is valid for $\text{conv}(S)$. (Hint: Consider the case $x \leq \lfloor b \rfloor$ and the case $x \geq \lfloor b \rfloor + 1$ separately.)

- (iii) Add this inequality to your drawing in Part (i).

- (b) Consider the polyhedron $P = \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^p : ax + gy \leq b\}$ and the mixed-integer set $T = P \cap (\mathbb{Z}_+^n \times \mathbb{R}_+^p)$. Let $f_0 = b - \lfloor b \rfloor$ and $f_j = a_j - \lfloor a_j \rfloor$.

- (i) Argue that the inequality

$$\sum_{j: f_j \leq f_0} \lfloor a_j \rfloor x_j + \sum_{j: f_j > f_0} a_j x_j + \sum_{j: g_j < 0} g_j y_j \leq b$$

is valid for P .

- (ii) Define

$$w := \sum_{j: f_j \leq f_0} \lfloor a_j \rfloor x_j + \sum_{j: f_j > f_0} \lceil a_j \rceil x_j$$

and

$$z := - \sum_{j: g_j < 0} g_j y_j + \sum_{j: f_j > f_0} (1 - f_j) x_j.$$

Show that $w - z \leq b$.

(iii) Apply Part (ii) of (a) to $\{(w, z) \in \mathbb{Z} \times \mathbb{R}_+ : w - z \leq b\}$ to show that the following inequality is valid for $\text{conv}(\mathbb{T})$:

$$\sum_{j=1}^n \left(\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0} \right) x_j + \frac{1}{1 - f_0} \sum_{j: g_j < 0} g_j y_j \leq \lfloor b \rfloor. \quad (1)$$

Here, for a scalar α , $(\alpha)^+ = \max\{\alpha, 0\}$.

(c) Without using any knowledge on split cuts, show that the GMI inequality obtained by adding a slack variable s to $ax + gy \leq b$ is identical to inequality (1)

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