15.083: Integer Programming and Combinatorial Optimization
Problem Set 3 Solutions

Due 9/30/2009

Problem (3.13)

(a) $\emptyset \in \mathcal{I}$ since $\emptyset \in \mathcal{I}_1$and $\emptyset \in \mathcal{I}_2$
Let $T = J_1 \cup J_2 \in \mathcal{I}$. For any $T' \subset T$ we construct $J_i' = J_i \backslash ((T \backslash T') \cap N_i)$ for $i = 1, 2$. We have $J_i' \in \mathcal{I}_i$ since M_i is a matroid. Thus $T' = J_1' \cup J_2' \in \mathcal{I}$
Let $T \subseteq N$. Since $N_1 \cap N_2 = \emptyset$, any basis, $B = J_1 \cup J_2$, of T is maximal iff J_1 and J_2 are maximal wrt their corresponding independence systems. Thus every maximal independent set in T has the same cardinality.

(b) $\emptyset \in \mathcal{I}$ trivially
For any $T' \subseteq T \in \mathcal{I}$, we have $|T' \cap N_i| \leq |T \cap N_i| \leq 1$, thus $T' \in \mathcal{I}$
For any $T \subseteq N$ every maximal basis of T has exactly one element in each partition that T covers, so all maximal independent sets have the same cardinality

(c) Create the set $\bar{N} = N \times \{0, 1\}$ let $N_1 = \{(n, 0) \in \bar{N}\}, N_2 = \{(n, 1) \in \bar{N}\}$, let $\mathcal{F}_1 = \{F \times \{0\} : F \in \mathcal{F}\}$, $\mathcal{F}_2 = \{F \times \{1\} : F \in \mathcal{F}\}$. With $(N_1, \mathcal{F}_1), (N_2, \mathcal{F}_2)$ we satisfy the assumptions of a) and can create a matroid M_1 of the form discussed in a). Next we form a partition matroid M_2 over $\bar{N}$ be taking the partitions $\{(v, 0), (v, 1)\} \forall v \in N$. S is of the desired form if and only if it is the projection of an element in the intersection of these two matroids. We can model this explicitly as an optimization problem over a polymatroid intersection polyhedron by creating the set $\bar{S} = S \times \{0, 1\}$ and solving the problem:

$$\boxed{\begin{array}{llll} \max \sum_v x_v & & & \\ \text{subject to} & & & \\ \sum_{v \in \bar{S}'} x_v & \leq & r_1(\bar{S}') & \forall \bar{S}' \subseteq \bar{S} \\ \sum_{v \in \bar{S}'} x_v & \leq & r_2(\bar{S}') & \forall \bar{S}' \subseteq \bar{S} \\ x_v & \in & \mathbb{Z}_+ & \forall v \in \bar{S} \end{array}}$$

This polymatroid intersection problem with have objective function value $|S|$ if and only if S can be partitioned as desired.

Problem (3.14) Hamitonian path is the intesection of a forest matroid on the underlying directed graph, a partition matroid over the outgoing arcs of each node, and a partition matroid over the incoming arcs of each node.

Problem (3.15)

(a) Define $x(S) = P\left(\left(\bigcap_{i \in S} A_i\right) \cap \left(\bigcap_{i \notin S} A_i^c\right)\right)$. We then can model our problem as follows:

$$
\begin{array}{|l|}
\hline
\min x(\emptyset) \\
\text{subject to} \\
\begin{array}{rcll}
\displaystyle\sum_{S:i\in S} x(S) & = & p_i & \forall i \in N \\
\displaystyle\sum_{S:i,j\in S} x(S) & \geq & p_{ij} & \forall (i,j) \in E \\
\displaystyle\sum_{S\subseteq N} x(S) & = & 1 & \\
x(S) & \geq & 0 & \forall S \subseteq N
\end{array} \\
\hline
\end{array}
$$

(b) The dual has the following form:

$$
\begin{array}{|l|}
\hline
\max \displaystyle\sum_{i\in N} p_i u_i + \sum_{(i,j)\in E} p_{ij} y_{ij} + z \\
\text{subject to} \\
\begin{array}{rcll}
\displaystyle\sum_{i\in S} u_i + \sum_{(i,j)\in E:i,j\in S} y_{ij} + z & \leq & 0 & \forall S \subseteq N, S \neq \emptyset \\
z & \leq & 1 & \\
y_{ij} & \geq & 0 & \forall (i,j) \in E
\end{array} \\
\hline
\end{array}
$$

Weak duality gives us that $P\left(\bigcap_{i\in N} A_i^c\right) = x(\emptyset) \geq \sum_{i\in N} p_i u_i + \sum_{(i,j)\in E} p_{ij} y_{ij} + z$ for all dual feasible solutions.

Hence if we pick $z = 1$ and $u_i = -1 \forall i \in N$, and maximize over the variables y_{ij}, we obtain:

$P\left(\bigcap_{i\in N} A_i^c\right) \geq 1 - \sum_{i\in N} p_i + Z$ where:

$$
\begin{array}{|l|}
\hline
Z = \max \displaystyle\sum_{(i,j)\in E} p_{ij} y_{ij} \\
\text{subject to} \\
\begin{array}{rcll}
\displaystyle\sum_{(i,j)\in E:i,j\in S} y_{ij} & \leq & |S| - 1 & \forall S \subseteq N, S \neq \emptyset \\
y_{ij} & \geq & 0 & \forall (i,j) \in E
\end{array} \\
\hline
\end{array}
$$

which is the maximum forest problem on the graph (N, E).

Problem (3.19) The dual problem is given by:

$$
\begin{array}{|l|}
\hline
\max \displaystyle\sum_{i\in V} p_i b_i - \sum_{(i,j)\in A} p_{ij} u_{ij} \\
\text{subject to} \\
\begin{array}{rcll}
p_{ij} & \geq & p_i - p_j - c_{ij} \forall (i,j) \in A & \\
p_{ij} & \geq & 0 & \forall (i,j) \in A
\end{array} \\
\hline
\end{array}
$$

Let p^* be an optimal dual solution. Note that since $u \geq 0$ we must have $p_{ij}^* = (p_i^* - p_j^* - c_{ij})^+$. We use the following randomized rounding procedure: generate a uniform random number on $[0,1]$ U and set $p_i = \lfloor p_i^* \rfloor$ if $p_i^* - \lfloor p_i^* \rfloor \leq U$ or $p_i = \lfloor p_i^* \rfloor + 1$ otherwise. [redacted]. We then set $p_{ij} = 0$ if $p_{ij}^* = 0$ or $p_{ij} = p_i - p_j - c_{ij}$ otherwise. That $E[\sum_{i\in V} p_i b_i - \sum_{(i,j)\in A} p_{ij} u_{ij}] = \sum_{i\in V} p_i^* b_i - \sum_{(i,j)\in A} p_{ij}^* u_{ij}$ is trivial, so we need only show that our integer solution p is dual feasible. Checking the following 8 cases exploiting the fact that $p_i - p_j - c_{ij}$ is integral shows feasibility:
$\{p_{ij}^* = 0, p_{ij}^* \geq 0\} \times \{p_i = \lfloor p_i^* \rfloor, p_i = \lfloor p_i^* \rfloor + 1\} \times \{p_j = \lfloor p_j^* \rfloor, p_j = \lfloor p_j^* \rfloor + 1\}$.

Problem (3.21) We consider the union of the set of breakpoints of all men for all women and call this set S. S divides $[0,1]$ into some number k of disjoint partitions with length p_i, $i = 1, ..., k$. If our random number U falls in the i^{th} partition, it generates the i^{th} stable matching x^i. Each of these matchings is distinct by construction. We have $x = \sum_i p_i x^i$ with $\sum_i p_i = 1$.

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